MINIMIZATION OF l_2-SENSITIVITY FOR 2-D SEPARABLE-DENOMINATOR STATE-SPACE DIGITAL FILTERS SUBJECT TO l_2-SCALING CONSTRAINTS USING A LAGRANGE FUNCTION AND A BISECTION METHOD

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Abstract: The problem of minimizing l_2-sensitivity subject to l_2-scaling constraints for two-dimensional (2-D) separable-denominator state-space digital filters is investigated. The coefficient sensitivity of the filter is analized by using a pure l_2-norm. An iterative algorithm for minimizing an l_2-sensitivity measure subject to l_2-scaling constraints is then explored by introducing a Lagrange function and utilizing an efficient bisection method. A numerical example is also presented to illustrate the utility of the proposed technique.

1 INTRODUCTION

In the fixed-point finite-word-length (FWL) implementation of recursive digital filters, the characteristics of an actual transfer function deviate from the original ones due to either truncation or rounding of filter coefficients. So far, several techniques for synthesizing two-dimensional (2-D) filter structures with low coefficient sensitivity have been reported (Kawamata et al., 1987)-(Hinamoto and Sugie, 2002). Some of them use a sensitivity measure evaluated by a mixture of l_1/l_2-norms (Kawamata et al., 1987; Hinamoto et al., 1992; Hinamoto and Takao, 1992), while the others rely on the use of a pure l_2-norm (Li, 1998; Hinamoto et al., 2002; Hinamoto and Sugie, 2002). Moreover, minimization of frequency-weighted sensitivity for 2-D state-space digital filters has been considered in accordance with both a mixed l_1/l_2-sensitivity measure and a pure l_2-sensitivity measure (Hinamoto et al., 1999). The l_2-sensitivity minimization is more natural and reasonable than the conventional l_1/l_2-mixed sensitivity minimization, but it is technically more challenging. Alternatively, a state-space digital filter with l_2-scaling constraints is beneficial for suppressing overflow oscillations (Mullis and Roberts, 1976; Hwang, 1977). However, satisfactory solution methods for l_2-sensitivity minimization subject to l_2-scaling constraints are still needed (Hinamoto et al., 2004; Hinamoto et al., 2005).

In this paper, an l_2-sensitivity minimization problem subject to l_2-scaling constraints for 2-D separable-denominator digital filters is formulated. An efficient iterative algorithm is explored to solve the constrained optimization problem directly. This is performed by applying a Lagrange function and an efficient bisection method. Computer simulation results by a numerical example demonstrate the validity and effectiveness of the proposed technique.

2 SENSITIVITY ANALYSIS

There is no loss of generality in assuming that a 2-D digital filter which is separable in the denominator can be described by the Roesser local state-space (LSS) model $\{A_1, A_2, A_4, b_1, b_2, c_1, c_2, d\}_{m+n}$ (Roesser, 1975; Hinamoto, 1980) as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \mathbf{0} & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(i,j)$$
$$y(i,j) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + d\,u(i,j) \tag{1}$$

where $x^h(i,j)$ is an $m \times 1$ horizontal state vector, $x^v(i,j)$ is an $n \times 1$ vertical state vector, $u(i,j)$ is a scalar input, $y(i,j)$ is a scalar output, and A_1, A_2, A_4,

b_1, b_2, c_1, c_2, and d are real constant matrices of appropriate dimensions. The LSS model in (1) is assumed to be asymptotically stable, separately locally controllable and separately locally observable (Kung et al., 1977). The transfer function of the LSS model in (1) is given by

$$\begin{aligned} H(z_1,z_2) &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} z_1 I_m - A_1 & -A_2 \\ \mathbf{0} & z_2 I_n - A_4 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + d \\ &= \begin{bmatrix} 1 & c_1(z_1 I_m - A_1)^{-1} \end{bmatrix} \\ &\qquad \cdot \begin{bmatrix} d & c_2 \\ b_1 & A_2 \end{bmatrix} \begin{bmatrix} 1 \\ (z_2 I_n - A_4)^{-1} b_2 \end{bmatrix}. \end{aligned} \tag{2}$$

Definition 1 : Let X be an $m \times n$ real matrix and let $f(X)$ be a scalar complex function of X, differentiable with respect to all the entries of X. The sensitivity function of f with respect to X is then defined as

$$S_X = \frac{\partial f}{\partial X} \quad \text{with} \quad (S_X)_{ij} = \frac{\partial f}{\partial x_{ij}} \tag{3}$$

where x_{ij} denotes the (i,j)th entry of the matrix X.

With these notations, it is easy to show that

$$\begin{aligned} \frac{\partial H(z_1,z_2)}{\partial A_1} &= Q^T(z_1) F^T(z_1,z_2) \\ \frac{\partial H(z_1,z_2)}{\partial A_2} &= Q^T(z_1) P^T(z_2) \\ \frac{\partial H(z_1,z_2)}{\partial A_4} &= G^T(z_1,z_2) P^T(z_2) \\ \frac{\partial H(z_1,z_2)}{\partial b_1} &= Q^T(z_1) \\ \frac{\partial H(z_1,z_2)}{\partial b_2} &= G^T(z_1,z_2) \\ \frac{\partial H(z_1,z_2)}{\partial c_1^T} &= F(z_1,z_2) \\ \frac{\partial H(z_1,z_2)}{\partial c_2^T} &= P(z_2) \end{aligned} \tag{4}$$

where

$$\begin{aligned} F(z_1,z_2) &= (z_1 I_m - A_1)^{-1}[b_1 + A_2 P(z_2)] \\ G(z_1,z_2) &= [c_2 + Q(z_1) A_2](z_2 I_n - A_4)^{-1} \\ P(z_2) &= (z_2 I_n - A_4)^{-1} b_2, \quad Q(z_1) = c_1 (z_1 I_m - A_1)^{-1}. \end{aligned}$$

The term d and the sensitivity with respect to it are coordinate independent, therefore they are neglected here.

Definition 2 : Let $X(z_1,z_2)$ be an $m \times n$ complex matrix valued function of the complex variables z_1 and z_2. The l_p-norm of $X(z_1,z_2)$ is then defined as

$$||X||_p = \left[\frac{1}{(2\pi j)^2} \oint \oint_{\Gamma^2} ||X(z_1,z_2)||_F^p \frac{dz_1 dz_2}{z_1 z_2} \right]^{1/p} \tag{5}$$

where $||X(z_1,z_2)||_F$ is the Frobenius norm of the matrix $X(z_1,z_2)$ defined by

$$||X(z_1,z_2)||_F = \left[\sum_{p=1}^{m} \sum_{q=1}^{n} \left| x_{pq}(z_1,z_2) \right|^2 \right]^{1/2}.$$

The overall l_2-sensitivity measure is now defined by

$$\begin{aligned} M_2 = & \left|\left| \frac{\partial H(z_1,z_2)}{\partial A_1} \right|\right|_2^2 + \left|\left| \frac{\partial H(z_1,z_2)}{\partial A_4} \right|\right|_2^2 \\ & + \left|\left| \frac{\partial H(z_1,z_2)}{\partial b_1} \right|\right|_2^2 + \left|\left| \frac{\partial H(z_1,z_2)}{\partial b_2} \right|\right|_2^2 \\ & + \left|\left| \frac{\partial H(z_1,z_2)}{\partial c_1^T} \right|\right|_2^2 + \left|\left| \frac{\partial H(z_1,z_2)}{\partial c_2^T} \right|\right|_2^2 \\ & + \left|\left| \frac{\partial H(z_1,z_2)}{\partial A_2} \right|\right|_2^2. \end{aligned} \tag{6}$$

From (4)-(6), it follows that

$$\begin{aligned} M_2 = & \,\mathrm{tr}\left[M_{A_1} + M_{A_4} + W^h + W^v + K^h + K^v \right] \\ & + \mathrm{tr}[W^h]\mathrm{tr}[K^v] \end{aligned} \tag{7}$$

where

$$\begin{aligned} M_{A_1} &= \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} [F(z_1^{-1}, z_2^{-1}) Q(z_1^{-1})] \\ &\qquad \cdot [Q^T(z_1) F^T(z_1,z_2)] \frac{dz_1 dz_2}{z_1 z_2} \\ M_{A_4} &= \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} [G^T(z_1,z_2) P^T(z_2)] \\ &\qquad \cdot [P(z_2^{-1}) G(z_1^{-1}, z_2^{-1})] \frac{dz_1 dz_2}{z_1 z_2} \end{aligned}$$

$$\begin{aligned} K^h &= \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} F(z_1,z_2) F^*(z_1,z_2) \frac{dz_1 dz_2}{z_1 z_2} \\ K^v &= \frac{1}{2\pi j} \oint_{|z_2|=1} P(z_2) P^*(z_2) \frac{dz_2}{z_2} \\ W^h &= \frac{1}{2\pi j} \oint_{|z_1|=1} Q^*(z_1) Q(z_1) \frac{dz_1}{z_1} \\ W^v &= \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} G^*(z_1,z_2) G(z_1,z_2) \frac{dz_1 dz_2}{z_1 z_2}. \end{aligned}$$

The matrices $K = K^h \oplus K^v$ and $W = W^h \oplus W^v$ are called the local controllability Gramian and local obsevability Gramian, respectively, and can be obtained by solving the following Lyapunov equations (Kawamata and Higuchi, 1986):

$$\begin{aligned} K^v &= A_4 K^v A_4^T + b_2 b_2^T \\ K^h &= A_1 K^h A_1^T + A_2 K^v A_2^T + b_1 b_1^T \\ W^h &= A_1^T W^h A_1 + c_1^T c_1 \\ W^v &= A_4^T W^v A_4 + A_2^T W^h A_2 + c_2^T c_2. \end{aligned} \tag{8}$$

Apply the following eigenvalue-eigenvector decompositions:

$$K^v = \sum_{i=1}^{n} \sigma_i^v u_i u_i^T, \qquad W^h = \sum_{i=1}^{m} \sigma_i^h v_i v_i^T \tag{9}$$

where σ_i^v and u_i (σ_i^h and v_i) are the ith eigenvalue and eigenvector of K^v (W^h), respectively. Then, we can write (7) as (Hinamoto and Sugie, 2002)

$$\begin{aligned} M_2 = &\sum_{i=0}^{n} \sigma_i^v \mathrm{tr}[W_i^h(I_m)] + \sum_{i=0}^{m} \sigma_i^h \mathrm{tr}[K_i^v(I_n)] \\ &+ \mathrm{tr}[W^h + W^v + K^h + K^v] + \mathrm{tr}[W^h]\mathrm{tr}[K^v] \end{aligned} \tag{10}$$

where $\sigma_0^v = \sigma_0^h = 1$,

$$\tilde{u}_i = \begin{cases} b_1 & \text{for } i = 0 \\ A_2 u_i & \text{for } i \geq 1 \end{cases}$$

$$\tilde{v}_i = \begin{cases} c_2^T & \text{for } i = 0 \\ A_2^T v_i & \text{for } i \geq 1 \end{cases}$$

and an $m \times m$ matrix $W_i^h(P_1)$ and an $n \times n$ matrix $K_i^v(P_4)$ are obtained by solving the following Lyapunov equations:

$$\begin{bmatrix} W_i^h(P_1) & * \\ * & * \end{bmatrix} = \begin{bmatrix} A_1 & \tilde{u}_i c_1 \\ \mathbf{0} & A_1 \end{bmatrix} \begin{bmatrix} W_i^h(P_1) & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} A_1 & \tilde{u}_i c_1 \\ \mathbf{0} & A_1 \end{bmatrix}^T + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_1 \end{bmatrix}$$

$$\begin{bmatrix} K_i^v(P_4) & * \\ * & * \end{bmatrix} = \begin{bmatrix} A_4 & \mathbf{0} \\ b_2 \tilde{v}_i^T & A_4 \end{bmatrix}^T \begin{bmatrix} K_i^v(P_4) & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} A_4 & \mathbf{0} \\ b_2 \tilde{v}_i^T & A_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_4^{-1} \end{bmatrix}.$$

3 SENSITIVITY MINIMIZATION

3.1 Problem Formulation

The following class of state-space coordinate transformations can be used without affecting the input-output map:

$$\begin{bmatrix} \overline{x}^h(i,j) \\ \overline{x}^v(i,j) \end{bmatrix} = \begin{bmatrix} T_1 & \mathbf{0} \\ \mathbf{0} & T_4 \end{bmatrix}^{-1} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{11}$$

where T_1 and T_4 are $m \times m$ and $n \times n$ nonsingular constant matrices, respectively. Performing this coordinate transformation to the LSS model in (1) yields a new realization $\{\overline{A}_1, \overline{A}_2, \overline{A}_4, \overline{b}_1, \overline{b}_2, \overline{c}_1, \overline{c}_2, d\}_{m+n}$ characterized by

$$\begin{aligned} &\overline{A}_1 = T_1^{-1} A_1 T_1, \qquad \overline{A}_2 = T_1^{-1} A_2 T_4 \\ &\overline{A}_4 = T_4^{-1} A_4 T_4, \qquad \overline{b}_1 = T_1^{-1} b_1 \\ &\overline{b}_2 = T_4^{-1} b_2, \qquad \overline{c}_1 = c_1 T_1, \qquad \overline{c}_2 = c_2 T_4 \\ &\overline{K}^h = T_1^{-1} K^h T_1^{-T}, \quad \overline{K}^v = T_4^{-1} K^v T_4^{-T} \\ &\overline{W}^h = T_1^T W^h T_1, \qquad \overline{W}^v = T_4^T W^v T_4. \end{aligned} \tag{12}$$

For the new realization, the l_2-sensitivity measure M_2 in (10) is changed to

$$\begin{aligned} M_2(P) = &\sum_{i=0}^{n} \sigma_i^v \mathrm{tr}[W_i^h(P_1) P_1^{-1}] + \sum_{i=0}^{m} \sigma_i^h \mathrm{tr}[K_i^v(P_4) P_4] \\ &+ \mathrm{tr}[W^h P_1 + W^v P_4 + K^h P_1^{-1} + K^v P_4^{-1}] \\ &+ \mathrm{tr}[W^h P_1]\mathrm{tr}[K^v P_4^{-1}] \end{aligned} \tag{13}$$

where $P = P_1 \oplus P_4$ and $P_i = T_i T_i^T$ for $i = 1, 4$.

If l_2-norm dynamic-range scaling constraints are imposed on the new local state vector $[\overline{x}^h(i,j)^T, \overline{x}^v(i,j)^T]^T$, then

$$\begin{aligned} (\overline{K}^h)_{ii} &= (T_1^{-1} K^h T_1^{-T})_{ii} = 1 \\ (\overline{K}^v)_{jj} &= (T_4^{-1} K^v T_4^{-T})_{jj} = 1 \end{aligned} \tag{14}$$

are required for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

From the above arguments, the problem is now formulated as follows: For given A_1, A_2, A_4, b_1, b_2, c_1 and c_2, obtain an $(m+n) \times (m+n)$ nonsingular matrix $T = T_1 \oplus T_4$ which minimizes (13) subject to l_2-scaling constraints in (14).

3.2 Problem Solution

If we sum up m constraints and n constraints in (14) separately, then we have

$$\mathrm{tr}[K^h P_1^{-1}] = m, \qquad \mathrm{tr}[K^v P_4^{-1}] = n. \tag{15}$$

Consequently, the problem of minimizing $M_2(P)$ in (13) subject to the constraints in (14) can be *relaxed* into the problem

$$\begin{aligned} &\text{minimize } M_2(P) \text{ in (13)} \\ &\text{subject to } \mathrm{tr}[K^h P_1^{-1}] = m \text{ and } \mathrm{tr}[K^v P_4^{-1}] = n. \end{aligned} \tag{16}$$

In order to solve (16), we define a Lagrange function of the problem as

$$\begin{aligned} J(P, \lambda_1, \lambda_4) = M_2(P) &+ \lambda_1(\mathrm{tr}[K^h P_1^{-1}] - m) \\ &+ \lambda_4(\mathrm{tr}[K^v P_4^{-1}] - n) \end{aligned} \tag{17}$$

where λ_1 and λ_4 are Lagrange multipliers. It is well known that the solution of problem (16) must satisfy the Karush-Kuhn-Tucker (KKT) conditions $\partial J(P,\lambda_1,\lambda_4)/\partial P_i = \mathbf{0}$ for $i = 1,4$ where the gradients are found to be

$$\begin{aligned}\frac{\partial J(P,\lambda_1,\lambda_4)}{\partial P_1} &= F_1(P) - P_1^{-1}F_2(P_1,\lambda_1)P_1^{-1}\\ \frac{\partial J(P,\lambda_1,\lambda_4)}{\partial P_4} &= F_3(P_4) - P_4^{-1}F_4(P,\lambda_4)P_4^{-1}\end{aligned} \tag{18}$$

with

$$\begin{aligned}F_1(P) &= \sum_{i=0}^{n} \sigma_i^v K_i^h(P_1) + (1+\mathrm{tr}[K^v P_4^{-1}])W^h\\ F_2(P_1,\lambda_1) &= \sum_{i=0}^{n} \sigma_i^v W_i^h(P_1) + (\lambda_1+1)K^h\\ F_3(P_4) &= \sum_{i=0}^{m} \sigma_i^h K_i^v(P_4) + W^v\\ F_4(P,\lambda_4) &= \sum_{i=0}^{m} \sigma_i^h W_i^v(P_4) + (\lambda_4+1+\mathrm{tr}[W^h P_1])K^v\end{aligned}$$

$$\begin{bmatrix} K_i^h(P_1) & * \\ * & * \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{0} \\ \tilde{u}_i c_1 & A_1 \end{bmatrix}^T \begin{bmatrix} K_i^h(P_1) & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} A_1 & \mathbf{0} \\ \tilde{u}_i c_1 & A_1 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_1^{-1} \end{bmatrix}$$

$$\begin{bmatrix} W_i^v(P_4) & * \\ * & * \end{bmatrix} = \begin{bmatrix} A_4 & b_2\tilde{v}_i^T \\ \mathbf{0} & A_4 \end{bmatrix} \begin{bmatrix} W_i^v(P_4) & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} A_4 & b_2\tilde{v}_i^T \\ \mathbf{0} & A_4 \end{bmatrix}^T + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_4 \end{bmatrix}.$$

Hence the above KKT conditions become

$$\begin{aligned}P_1F_1(P)P_1 &= F_2(P_1,\lambda_1)\\ P_4F_3(P_4)P_4 &= F_4(P,\lambda_4).\end{aligned} \tag{19}$$

Two equations in (19) are highly nonlinear with respect to P_1 and P_4. An effective approach to solving two equations in (19) is to *relax* them into the following recursive second-order matrix equations:

$$\begin{aligned}P_1^{(i+1)}F_1(P^{(i)})P_1^{(i+1)} &= F_2(P_1^{(i)},\lambda_1^{(i+1)})\\ P_4^{(i+1)}F_3(P_4^{(i)})P_4^{(i+1)} &= F_4(P^{(i)},\lambda_4^{(i+1)})\end{aligned} \tag{20}$$

with the initial condition $P^{(0)} = P_1^{(0)} \oplus P_4^{(0)} = I_{m+n}$. The solutions $P_1^{(i+1)}$ and $P_4^{(i+1)}$ of (20) are given by

$$\begin{aligned}P_1^{(i+1)} &= F_1^{-\frac{1}{2}}(P^{(i)})[F_1^{\frac{1}{2}}(P^{(i)})F_2(P_1^{(i)},\lambda_1^{(i+1)})\\ &\quad \cdot F_1^{\frac{1}{2}}(P^{(i)})]^{\frac{1}{2}}F_1^{-\frac{1}{2}}(P^{(i)})\\ P_4^{(i+1)} &= F_3^{-\frac{1}{2}}(P_4^{(i)})[F_3^{\frac{1}{2}}(P_4^{(i)})F_4(P^{(i)},\lambda_4^{(i+1)})\\ &\quad \cdot F_3^{\frac{1}{2}}(P_4^{(i)})]^{\frac{1}{2}}F_3^{-\frac{1}{2}}(P_4^{(i)})\end{aligned} \tag{21}$$

respectively. Here, Lagrange multipliers $\lambda_1^{(i+1)}$ and $\lambda_4^{(i+1)}$ can be efficiently obtained using a bisection method so that

$$\begin{aligned}f_1(\lambda_1^{(i+1)}) &= m - \mathrm{tr}[\tilde{K}_h^{(i)}\tilde{F}_2^{(i)}(\lambda_1^{(i+1)})] = 0\\ f_4(\lambda_4^{(i+1)}) &= n - \mathrm{tr}[\tilde{K}_v^{(i)}\tilde{F}_4^{(i)}(\lambda_4^{(i+1)})] = 0\end{aligned} \tag{22}$$

are satisfied where

$$\begin{aligned}\tilde{K}_h^{(i)} &= F_1^{\frac{1}{2}}(P^{(i)})K^hF_1^{\frac{1}{2}}(P^{(i)})\\ \tilde{K}_v^{(i)} &= F_3^{\frac{1}{2}}(P_4^{(i)})K^vF_3^{\frac{1}{2}}(P_4^{(i)})\\ \tilde{F}_2^{(i)}(\lambda_1^{(i+1)}) &= [F_1^{\frac{1}{2}}(P^{(i)})F_2(P_1^{(i)},\lambda_1^{(i+1)})F_1^{\frac{1}{2}}(P^{(i)})]^{-\frac{1}{2}}\\ \tilde{F}_4^{(i)}(\lambda_4^{(i+1)}) &= [F_3^{\frac{1}{2}}(P_4^{(i)})F_4(P^{(i)},\lambda_4^{(i+1)})F_3^{\frac{1}{2}}(P_4^{(i)})]^{-\frac{1}{2}}.\end{aligned}$$

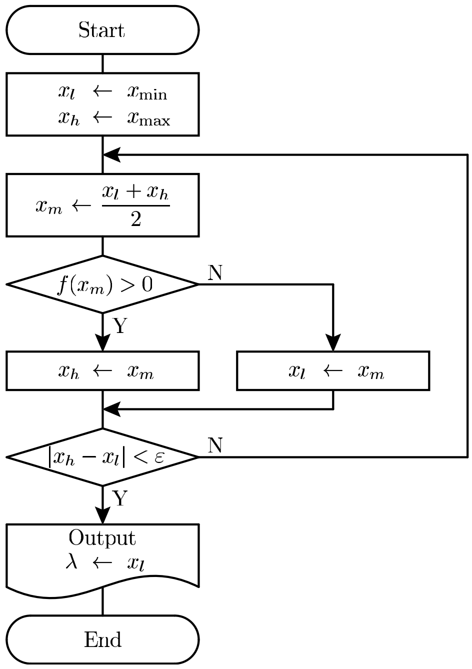

Figure 1: A flow chart of the bisection method.

A flow chart of the above bisection method is shown in Fig. 1. The iteration process continues until

$$|J(P^{(i+1)},\lambda_1^{(i+1)},\lambda_4^{(i+1)}) - J(P^{(i)},\lambda_1^{(i)},\lambda_4^{(i)})| < \varepsilon \tag{23}$$

is satisfied for a prescribed tolerance $\varepsilon > 0$. If the iteration is terminated at step i, then $P^{(i)}$ is viewed as a solution point.

Once positive-definite symmetric matrices P_1 and P_4 satisfying $\mathrm{tr}[K_1P_1^{-1}] = m$ and $\mathrm{tr}[K_4P_4^{-1}] = n$ were obtained, it is possible to construct an $m \times m$ orthogonal matrix U_1 and an $n \times n$ orthogonal matrix U_4 so that matrix $T = P_1^{1/2}U_1 \oplus P_4^{1/2}U_4$ satisfies L_2-scaling constraints in (14). (Hinamoto et al., 2005)

4 ILLUSTRATIVE EXAMPLE

Suppose that a 2-D separable-denominator digital filter $\{A_1^o, A_2^o, A_4^o, b_1^o, b_2^o, c_1^o, c_2^o, d\}_{3+3}$ in (1) is specified by

$$A_1^o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.599655 & -1.836929 & 2.173645 \end{bmatrix}$$

$$A_2^o = \begin{bmatrix} 0.064564 & 0.033034 & 0.012881 \\ 0.091213 & 0.110512 & 0.102759 \\ 0.097256 & 0.151864 & 0.172460 \end{bmatrix}$$

$$A_4^o = \begin{bmatrix} 0 & 0 & 0.564961 \\ 1 & 0 & -1.887939 \\ 0 & 1 & 2.280029 \end{bmatrix}$$

$$b_1^o = \begin{bmatrix} 0.047053 \\ 0.062274 \\ 0.060436 \end{bmatrix}, \qquad b_2^o = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1^o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$c_2^o = \begin{bmatrix} 0.016556 & 0.012550 & 0.008243 \end{bmatrix}$$

$$d = 0.019421.$$

By performing the l_2-scaling for the above LSS model with a diagonal coordinate-transformation matrix $T^o = T_1^o \oplus T_4^o$ where

$$T_1^o = \mathrm{diag}\{0.992289, 0.987696, 0.964582\}$$

$$T_4^o = \mathrm{diag}\{4.636056, 10.980193, 8.012802\}$$

we obtained

$$A_1 = \begin{bmatrix} 0.000000 & 0.995371 & 0.000000 \\ 0.000000 & 0.000000 & 0.976599 \\ 0.616880 & -1.880945 & 2.173645 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.301648 & 0.365538 & 0.104015 \\ 0.428136 & 1.228560 & 0.833645 \\ 0.467440 & 1.728723 & 1.432628 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0.000000 & 0.000000 & 0.976460 \\ 0.422220 & 0.000000 & -1.377725 \\ 0.000000 & 1.370331 & 2.280029 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 0.047419 & 0.063050 & 0.062655 \end{bmatrix}^T$$

$$b_2 = \begin{bmatrix} 0.215701 & 0.000000 & 0.000000 \end{bmatrix}^T$$

$$c_1 = \begin{bmatrix} 0.992289 & 0.000000 & 0.000000 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} 0.076755 & 0.137801 & 0.066050 \end{bmatrix}$$

and the l_2-sensitivity of the scaled LSS model was found to be

$$M_2 = 4526.0790.$$

Choosing $P^{(0)} = P_1^{(0)} \oplus P_4^{(0)} = I_6$ in (21) as initial estimate, $x_{min} = -2^{20}$ and $x_{max} = 2^{20}$ in the bisection method, and tolerance $\varepsilon = 10^{-8}$ in Fig. 1 and (23), it took the proposed algorithm 15 iterations to converge to the solution $P^{opt} = P_1^{opt} \oplus P_4^{opt}$ where

$$P_1^{opt} = \begin{bmatrix} 0.992455 & 0.702756 & 0.373871 \\ 0.702756 & 0.724033 & 0.597920 \\ 0.373871 & 0.597920 & 0.674661 \end{bmatrix}$$

$$P_4^{opt} = \begin{bmatrix} 2.200512 & -2.005367 & 1.676709 \\ -2.005367 & 1.913721 & -1.647192 \\ 1.676709 & -1.647192 & 1.480797 \end{bmatrix}$$

or equivalently, $T^{opt} = T_1^{opt} \oplus T_4^{opt}$ where

$$T_1^{opt} = \begin{bmatrix} -0.975337 & -0.066061 & 0.191859 \\ -0.619458 & 0.147201 & 0.564479 \\ -0.291519 & 0.450550 & 0.621839 \end{bmatrix}$$

$$T_4^{opt} = \begin{bmatrix} -0.799684 & 0.585116 & -1.103928 \\ 0.493843 & -0.684596 & 1.095978 \\ -0.336031 & 0.804236 & -0.849167 \end{bmatrix}.$$

The minimized l_2-sensitivity measure in (17) corresponding to the above solution was found to be

$$J(P^{opt}, \lambda_1, \lambda_4) = 101.0064$$

with $\lambda_1 = 4.786834$ and $\lambda_4 = -4.094596$. By substituting $T = T^{opt}$ obtained above into (12), the optimal state-space filter structure that minimizes (13) subject to the l_2-scaling constraints in (14) was synthesized as

$$\overline{A}_1 = \begin{bmatrix} 0.694418 & -0.112298 & -0.412379 \\ -0.096981 & 0.765920 & -0.345179 \\ 0.282990 & 0.456524 & 0.713306 \end{bmatrix}$$

$$\overline{A}_2 = \begin{bmatrix} 0.138105 & -0.073790 & 0.140661 \\ -0.132057 & 0.634682 & -0.262494 \\ 0.158022 & -0.104957 & 0.516782 \end{bmatrix}$$

$$\overline{A}_4 = \begin{bmatrix} 0.699418 & -0.018435 & 0.273811 \\ -0.091049 & 0.837579 & 0.358967 \\ -0.257686 & -0.254075 & 0.743031 \end{bmatrix}$$

$$\overline{b}_1 = \begin{bmatrix} -0.038277 & 0.028296 & 0.062312 \end{bmatrix}^T$$

$$\overline{b}_2 = \begin{bmatrix} -0.758218 & 0.129041 & 0.422255 \end{bmatrix}^T$$

$$\overline{c}_1 = \begin{bmatrix} -0.967816 & -0.065551 & 0.190380 \end{bmatrix}$$

$$\overline{c}_2 = \begin{bmatrix} -0.015522 & 0.003691 & 0.010209 \end{bmatrix}$$

whose horizontal and vertical controllability Gramians were given by

$$K_{opt}^h = \begin{bmatrix} 1.000000 & -0.090933 & -0.400242 \\ -0.090933 & 1.000000 & 0.400242 \\ -0.400242 & 0.400242 & 1.000000 \end{bmatrix}$$

$$K_{opt}^v = \begin{bmatrix} 1.000000 & -0.126238 & -0.520618 \\ -0.126238 & 1.000000 & 0.520618 \\ -0.520618 & 0.520618 & 1.000000 \end{bmatrix}.$$

Profile of the l_2-sensitivity measure, and profile of the parameters λ_1 and λ_4 during the first 15 iterations of the proposed algorithm are shown in Figs. 2 and 3, respectively.

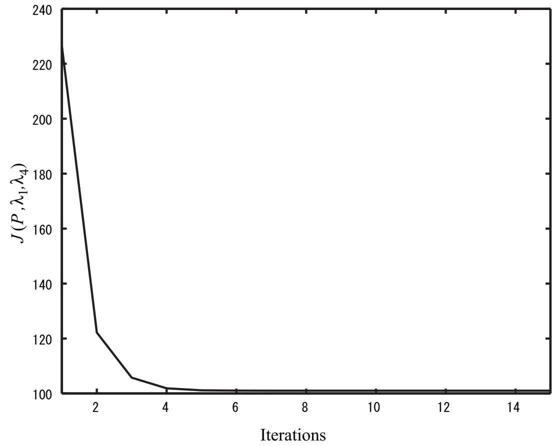

Figure 2: l_2-Sensitivity Performance.

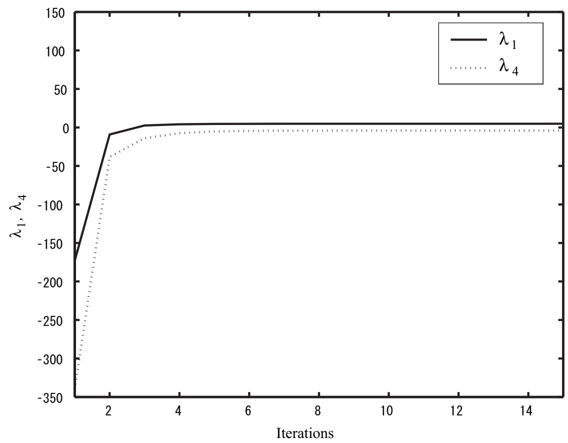

Figure 3: λ_1 and λ_4 Performances.

5 CONCLUSION

The problem of minimizing the l_2-sensitivity measure subject to l_2-scaling constraints for 2-D separable-denominator state-space digital filters has been formulated. An iterative method for minimizing l_2-sensitivity subject to l_2-scaling constraints has been explored. This has been performed by using a Lagrange function and an efficient bisection method. Computer simulation results have demonstrated the validity and effectiveness of the proposed technique.

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