IMPROVED ROBUSTNESS OF MULTIVARIABLE MODEL PREDICTIVE CONTROL UNDER MODEL UNCERTAINTIES

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Abstract: This paper presents a state-space methodology for enhancing the robustness of multivariable MPC controlled systems through the convex optimization of a multivariable Youla parameter. The procedure starts with the design of an initial stabilizing Model Predictive Controller in the state-space representation, which is then robustified under modeling errors considered as unstructured uncertainties. The resulting robustified MIMO control law is finally applied to the model of a stirred tank reactor to reduce the impact of measurement noise and modelling errors on the system.

1 INTRODUCTION

Model predictive control strategies are widely used in industrial applications, resulting in improved performance, with a practical implementation of the controller which remains simple. However, starting with a controller design based on a ‘nominal’ model of the system, the question of its robustness towards model uncertainties or disturbances acting on the system always occurs in an industrial environment.

Some methods in the literature deal with robustness maximisation, but in the transfer function formalism (Kouvaritakis et al., 1992), (Yoon and Clarke, 1995), (Dumur and Boucher, 1998), and mainly applied to SISO systems, which makes the generalization to multivariable systems much more complicated.

The purpose of this paper is to present a methodology enhancing the robustness of an initial MIMO predictive controller towards model uncertainties. The state-space design allows the robustification process to be handled in a convenient way. A two-step procedure is followed. An initial MIMO MPC controller is first designed, its robustness is then enhanced via the Youla parametrization, without significantly increasing the complexity of the final control law. The Youla parametrization allows formulating frequency constraints as convex optimization, the entire problem being solved with LMI (Linear Matrix Inequality) techniques.

The paper is organized as follows. Section 2 reminds the main steps leading to the MPC controller in the state-space representation. Section 3 gives the background material required to formulate the robustification strategy, from the Youla parametrization to the robustness criteria under unstructured uncertainties. The elaboration of the robustified controller in state-space representation for this type of uncertainties is further proposed in Section 4. Section 5 provides the application of this control strategy to a stirred tank reactor. Section 6 presents some conclusions and further perspectives.

2 MIMO MPC IN STATE-SPACE FORMULATION

This section focuses on the design of an initial MIMO MPC law. Compared to approaches proposed in the literature based on transfer function formalism, the state-space representation framework chosen here (Camacho and Bordons, 2004) leads to a simplified formulation and reduced computation efforts for MIMO systems. Consider the following discrete time MIMO LTI system:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \\
    y(k) &= Cx(k)
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) are the system state-space matrices, \( x \in \mathbb{R}^{n \times 1} \) describes the MIMO system states, \( u \in \mathbb{R}^{m \times 1} \) is the input vector and \( y \in \mathbb{R}^{p \times 1} \) is the output vector.

Next step is to add an integral action to this state-space representation which will guarantee cancellation of steady-state errors:

\[
u(k) = u(k-1) + \Delta u(k)
\]
This results in an increase of the system states as:
\[
\begin{align*}
\dot{x}_e(k+1) &= A_e x_e(k) + B_e \Delta u(k) \\
\psi(k) &= C_e x_e(k)
\end{align*}
\]
(3)
where the extended state-space representation \( x_e(k) = [x^T(k), u^T(k-1)]^T \) is characterized by:
\[
\begin{align*}
A_e &= \begin{bmatrix} A & B \\ 0_{m,r} & I_m \end{bmatrix}, \\
B_e &= \begin{bmatrix} B \\ I_m \end{bmatrix}, \\
C_e &= \begin{bmatrix} \mathbf{0} & 0_{p,m} \end{bmatrix}.
\end{align*}
\]

The control signal is derived by minimizing the following quadratic objective function:
\[
J = \sum_{i=N_1}^{N_2} \| \hat{y}(k+i) - w(k+i) \|_{Q_F(i)}^2 + \sum_{i=0}^{N_f-1} \| \Delta u(k+i) \|_{R_F(i)}^2
\]
(4)
where the future control increments \( \Delta u(k+i) \) are supposed to be zero for \( i \geq N_u \). The signal \( w \) represents the setpoint. It is assumed in further developments that the same output prediction horizons \((N_1, N_2)\) and the same control horizon \(N_u\) are applied for all input/output transfer functions. \( Q_F \) and \( R_F \) are weighting matrices. The predicted output vector has the following form:
\[
\hat{y}(k+i) = C A^i \hat{x}(k) + \sum_{j=0}^{i-1} C A^{i-j-1} B u(k+j)
\]
(5)
where the input vector can be written as:
\[
u(k+j) = u(k-1) + \sum_{i=0}^{j} \Delta u(k+i)
\]
(6)
The state estimate is derived from the observer:
\[
\hat{x}_e(k+1) = A_e \hat{x}_e(k) + B_e \Delta u(k) + K \psi(k) - C_e \dot{x}_e(k)
\]
(7)
The multivariable observer gain \( K \) is designed through a classical method of eigenvectors, arbitrarily placing the eigenvalues of \( A_e - K C_e \) in a stable region, as detailed in (Magni, 2002). The observer gain \( K \) is obtained from the extended state-space description and will be used for further mathematical calculation in the robustification procedure. However, this design aspect is not crucial since the convex robustification method should lead to an optimal set of these eigenvalues. Moreover, the input/output transfer function is not influenced by the eigenvalues placement used to find \( K \) (Boyd and Barratt, 1991). The objective function can be rewritten in the matrix formalism (Maciejowski, 2001):
\[
J = \| Y(k) - W(k) \|_{Q_F}^2 + \| \Delta U(k) \|_{R_F}^2
\]
(8)
where \( Q_F = \text{diag}(Q_F(N_1), \ldots, Q_F(N_2)) \),
\[
R_F = \text{diag}(R_F(0), \ldots, R_F(N_u - 1))
\]
\[
Y(k) = [\tilde{y}^T(k + N_1) \ldots \tilde{y}^T(k + N_2)]^T,
\]
\[
W(k) = [w^T(k + N_1) \ldots w^T(k + N_2)]^T,
\]
\[
\Delta U(k) = \Delta u^T(k) \ldots \Delta u^T(k + N_u - 1)
\]

Using these notations, the output vector \( Y(k) \) can be written in the following matrix form, with the definition of the vector \( \Theta(k) \) as a tracking error:
\[
Y(k) = \Psi \hat{x}(k) + \Phi u(k - 1) + \Phi_A \Delta U(k)
\]
(9)
\[
\Theta(k) = W(k) - \Psi \hat{x}(k) - \Phi u(k - 1)
\]
(10)
with \( \Psi = \left[(CA^{N_1})^T \ldots (CA^{N_2})^T\right]^T \),
\[
\Phi = \begin{bmatrix} \Sigma^T_{N_1 - 1} \ldots \Sigma^T_{N_2} \end{bmatrix}^T, \\
\Phi_A = \sum_{j=0}^{N_f - 1} A^{j-1} B, \\
\Sigma_j = (\Sigma_j)^T,
\]
\[
\Phi_A = \begin{bmatrix} \Sigma_{N_1-1} \ldots \Sigma_0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\Sigma_{N_2-1} \ldots \Sigma_{N_2-N_1} & \Sigma_{N_2-N_1-1} \ldots \Sigma_{N_2-N_u} \end{bmatrix}
\]
The objective function is now given by:
\[
J = \| \Phi_A \Delta U(k) - \Theta(k) \|_{Q_F}^2 + \| \Delta U(k) \|_{R_F}^2
\]
(11)

which analytical minimization provides:
\[
\Delta U(k) = (R_F + \Phi_A^T Q_F \Phi_A)^{-1} \Phi_A^T Q_F \Theta(k)
\]
(12)

Applying the receding horizon principle, only the first component of each future control sequence is applied to the system, meaning that the first \( m \) lines of \( \Delta U(k) \) are used:
\[
\Delta u(k) = \mu \Theta(k)
\]
(13)
with \( \mu = \left[ \mu_m, \mu_{m+m(N_u-1)} \right] (R_F + \Phi_A^T Q_F \Phi_A)^{-1} \Phi_A^T Q_F \).

The system model, the observer and the predictive control can be represented in the state-space formulation according to Figure 1. The control signal depends on the control gain \( L = [L_1, L_2] \) and the setpoint filter \( F_w \):
\[
\Delta u(k) = F_w \omega(k) - L \dot{x}_e(k)
\]
(14)
with \( L_1 = \mu \Psi, \ L_2 = \mu \Phi, \ F_w = \text{diag}(F_{w,1}, \ldots, F_{w,p}) \) related to the structure of \( \mu \) and \( \omega(k) \).
3 ROBUSTNESS USING THE YOULA PARAMETER

This section overviews a technique that improves the robustness of the previous multivariable MPC law in terms of the Youla parameter, also named \( Q \) parameter. Any stabilizing controller (Boyd and Barratt, 1991), (Maciejowski, 1989) can be represented by a state-space feedback controller coupled with an observer and a Youla parameter. This part focuses on the main steps leading to the multivariable \( Q \) parameter (here with \( p \) inputs and \( m \) outputs) that robustifies the MPC law described in Section 2.

3.1 Stabilizing Control Law

The whole class of stabilizing control law can be obtained from an initial stabilizing controller via the Youla parametrization. The first step considers additional inputs \( u' \) and outputs \( y' \) with a zero transfer between them (\( T_{zz} = 0 \) in Figure 2).

\[
\begin{bmatrix}
X_1 \\
\gamma
\end{bmatrix} > 0
\]

This expression can be transformed into a LMI, which variables are \( X_1 \), \( \gamma \) and the \( Q \) parameter included in the closed-loop matrices, as shown in (Clement and Duc, 2000). As a result, the optimization problem is formulated as the minimization of \( \gamma \) under this LMI constraint.

4 ROBUSTIFIED MIMO MPC

The previous robustification strategy based on the Youla parameter is now applied to an initial MIMO state-space MPC calculated as shown in Section 2. The robustness maximization under additive unstructured uncertainties is also equivalent to the minimization of the influence of a measurement noise \( b \) on the control signal \( u \) (Figure 4); the

\[
T_{zw} = T_{11} Q + T_{12} Q T_{21} \quad (15)
\]

where \( T_{11}, T_{12}, T_{21} \) depends on the input vector \( w \) and output vector \( z \) considered.

3.2 Robustness Under Frequency Constraints

Practical applications always deal with neglected dynamics and potential disturbances, so that robustness under unstructured uncertainties must be addressed as shown in Figure 3.

According to the small gain theorem (Maciejowski, 1989), robustness under unstructured uncertainties \( \Delta_u \) is maximized as:

\[
\min_Q \| T_{zw} W_T \|_{\infty} \quad (16)
\]

where the weighting term \( W_T \) reflects the frequency range where model uncertainties are more important. For multivariable systems, the \( H_\infty \) norm can be calculated as the maximum of the higher singular values. The following theorem formulates the previous \( H_\infty \) norm minimization.

Theorem (Clement and Duc, 2000) and (Boyd et al., 1994): A discrete time system given by the state-space representation \( (A_{cl}, B_{cl}, C_{cl}, D_{cl}) \) is stable and admits a \( H_\infty \) norm lower than \( \gamma \) if and only if:

\[
\begin{bmatrix}
-X_1^{-1} & A_{cl} & B_{cl} & 0 \\
A_{cl}^T & -X_1 & 0 & C_{cl}^T \\
B_{cl}^T & 0 & -\gamma I & D_{cl}^T \\
0 & C_{cl} & D_{cl} & -\gamma I
\end{bmatrix} < 0 \quad (17)
\]

This expression can be transformed into a LMI, which variables are \( X_1 \), \( \gamma \) and the \( Q \) parameter included in the closed-loop matrices, as shown in (Clement and Duc, 2000). As a result, the optimization problem is formulated as the minimization of \( \gamma \) under this LMI constraint.
transfer (15) between \( w \) and \( z \) corresponds to the transfer from \( b \) to \( u \). The \( H_\infty \) norm of this transfer will be further minimized using LMI tools.

\[
\begin{align*}
\Delta u(k) &= F_n w(k) - L_s x(k) - u'(k) \\
&= A_n x_n(k) + B_n u(k) + C_n x_n(k) + b(k)
\end{align*}
\] (18)

(19)

To calculate the closed-loop transfer function, the initial state is increased, adding the prediction error:

\[
\varepsilon(k) = x_e(k) - \hat{x}_e(k)
\] (20)

Considering only the terms related to \( b(k) \) as they are part of the minimization process, the following state-space system is derived:

\[
\begin{bmatrix}
x_e(k+1) \\
\varepsilon(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_n & A_{n1} \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
x_e(k) \\
\varepsilon(k)
\end{bmatrix} +
\begin{bmatrix}
0 & -B_n \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
b(k) \\
u(k)
\end{bmatrix}
\] (21)

\[
\begin{bmatrix}
y'(k) \\
\varepsilon'(k)
\end{bmatrix} =
\begin{bmatrix}
0 & C_n
\end{bmatrix}
\begin{bmatrix}
x_e(k) \\
\varepsilon(k)
\end{bmatrix} +
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
b(k) \\
u'(k)
\end{bmatrix}
\] (22)

with \( A_1 = A_n - B_n L \), \( A_2 = A_n - K C_n \), \( A_3 = B_n \).

According to the theory given in Section 3.1, the Youla parameter can be added to robustify the initial controller, since the transfer between \( y'(k) \) and \( u'(k) \) is zero (without measurement noise), the multivariable output \( y' \) depends only on \( \varepsilon(k) \), which is independent from \( x_e(k) \) and \( u'(k) \).

\[
\begin{bmatrix}
\dot{x}_e(k+1) \\
\dot{\varepsilon}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_n & A_{n1} \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
0 & -B_n \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}(k)
\end{bmatrix}
\] (23)

Incorporating the \( W_e \) weighting, a new extended state-space description can be emphasized:

\[
\begin{bmatrix}
\dot{x}_e(k+1) \\
\dot{\varepsilon}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_n & A_{n1} \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
0 & -B_n \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}(k)
\end{bmatrix}
\] (24)

\[
\begin{bmatrix}
y'(k) \\
\varepsilon'(k)
\end{bmatrix} =
\begin{bmatrix}
0 & C_n
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}'(k)
\end{bmatrix}
\] (25)

With \( x_e(k) = [x^T(k) \ u^T(k-1) \ x_e^T(k)]^T \), \( B_p = [B^T \ I \ B_e^T]^T \)

As described in Section 3.2, a multivariable Youla parameter \( Q \in \Re \) is added for robustification purposes leading to a convex optimization problem. Since this problem leads to a \( Q \) parameter which varies in the infinite-dimensional space \( \Re \), a sub-optimal solution considers for each input/output pairs \( (i, j) \) a finite-dimensional subspace generated by an orthonormal base of discrete stable transfer functions such as a polynomial or FIR filter:

\[
Q^{ij} = \sum_{l=0}^{n_Q} q_{ij}^{\ell} q^{l-1}
\] (26)

In the state-space formalism, this MIMO Youla parameter can be obtained using a fixed pair \((A_Q \in \Re_{mpQ \times mQ}, B_Q \in \Re_{m \times p}, \) and designing only the variable pair \((C_Q \in \Re_{m \times mpQ}, D_Q \in \Re_{m \times p})\):

\[
\begin{bmatrix}
\dot{x}_Q(k+1) \\
\dot{u}'(k)
\end{bmatrix} =
\begin{bmatrix}
A_Q & B_Q y(k) \\
C_Q x_Q(k) + D_Q y(k)
\end{bmatrix}
\] (27)

\[
\begin{bmatrix}
0_{n_Q \times n_Q -1} & 0 \\
1_{n_Q -1 \times 1} & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0_{n_Q -1 \times 1} \\
0_{n_Q -1 \times 1} & 1
\end{bmatrix},
\begin{bmatrix}
q_1^{ij} \\
q^{ij}_Q
\end{bmatrix},
\begin{bmatrix}
q^{ij}_Q \\
q_1^{ij}
\end{bmatrix}
\] (28)

With \( A_Q = \text{diag}(a_Q, \ldots, a_Q), B_Q = \text{diag}(b_Q, \ldots, b_Q) \).

Adding this Youla parameter leads to the following closed-loop state-space description:

\[
\begin{bmatrix}
\dot{x}_e(k+1) \\
\dot{\varepsilon}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_n & A_{n1} \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
0 & -B_n \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}(k)
\end{bmatrix}
\] (23)

Incorporating the \( W_e \) weighting, a new extended state-space description can be emphasized:

\[
\begin{bmatrix}
\dot{x}_e(k+1) \\
\dot{\varepsilon}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_n & A_{n1} \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
0 & -B_n \\
-K & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}(k)
\end{bmatrix}
\] (24)

\[
\begin{bmatrix}
y'(k) \\
\varepsilon'(k)
\end{bmatrix} =
\begin{bmatrix}
0 & C_n
\end{bmatrix}
\begin{bmatrix}
\dot{x}_e(k) \\
\dot{\varepsilon}(k)
\end{bmatrix} +
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{b}(k) \\
\dot{u}'(k)
\end{bmatrix}
\] (25)

With \( x_e(k) = [x^T(k) \ u^T(k-1) \ x_e^T(k)]^T \), \( B_p = [B^T \ I \ B_e^T]^T \)
The whole problem results in the minimization of \( \gamma \) subject to the LMI constraint (31):

\[
\min_{\gamma} \gamma
\]

subject to the LMI

\[
C_{el} = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 - D_n D_Q \overline{C}_e - D_n C_Q \end{bmatrix}, \quad D_{el} = -D_n D_Q,
\]

\[
A_{el} = \begin{bmatrix} \overline{A}_1 & \overline{A}_3 - \overline{B}_w D_Q \overline{C}_e - (\overline{B}_w C_Q \end{bmatrix}, \quad B_{el} = -B_w D_Q
\]

This state-space representation is the crucial point of the robustification method. With the result of the theorem in Section 3, the first step to transform (17) into a LMI consists in multiplying it to the right and to the left with positive definite matrices \( \Pi = \text{diag}(X_1, I, I, I) \) and \( \Pi^T \) as in (Clement and Duc, 2000). To overcome this problem, the following bijective substitution is introduced (Clement and Duc, 2000):

\[
\begin{bmatrix} -X_1 & X_1 A_{el} & X_1 B_{el} & 0 \\ A_{el}^T X_1 & -X_1 & C_{el}^T & 0 \\ B_{el}^T X_1 & 0 & -\gamma I & D_{el}^T \\ 0 & C_{el} & D_{el} & -\gamma I \end{bmatrix} < 0 \tag{29}
\]

which is not yet a LMI because terms such as \( X_1 A_{el} \) and \( X_1 B_{el} \) are not linear in \( X_1, C_Q \) and \( D_Q \). To overcome this problem, the following technical manipulations are introduced (Clement and Duc, 2000):

\[
R^{\text{non}} \rightarrow R^{\text{non}}
\]

\[
X = \begin{bmatrix} W_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} R_i^T S_{12}^T \mathbf{0} & R_i^T S_{11} \mathbf{0} & S_{11}^T T_{12} & S_{11}^T T_{11} & S_{12}^T T_{12} \end{bmatrix}
\tag{30}
\]

with \( R_i = W_i^{-1}, \quad S_i = -W_i^{-1} Z_i, \quad T_i = Y_i - Z_i^T W_i^{-1} Z_i \).

Next step to the LMI is to multiply (29) on the right with \( \Pi = \text{diag} \left( \frac{R_i^T}{S_i^T}, 1, 1, 1 \right) \) and on the left with \( \Pi^T \). After technical manipulations, the following LMI is obtained:

\[
\begin{bmatrix} -R_1 & 0 & 0 & X_r & X_{el} \end{bmatrix} \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix} < 0 \tag{31}
\]

where

\[
t_1 = \overline{A}_1 \overline{S}_1 + \overline{S}_1 \overline{A}_2 + \overline{S}_1 \overline{B}_w D_Q \overline{C}_e + \overline{A}_3 \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_2 = T_{11} \overline{A}_1 + T_{12} \overline{B}_w D_Q \overline{C}_e, \quad t_3 = T_{11} \overline{A}_2 + T_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_4 = \overline{A}_1 \overline{S}_{12} - \overline{S}_{12} \overline{A}_1 - \overline{B}_w D_Q \overline{C}_e, \quad t_5 = T_{12} \overline{A}_2 + T_{22} \overline{A}_Q - \overline{A}_3 \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_6 = -\overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_7 = -\overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_8 = -\overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_9 = -\overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{10} = \overline{A}_1 \overline{S}_1 + \overline{A}_2 + \overline{A}_3 \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{11} = T_{11} \overline{A}_1 + T_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{12} = T_{11} \overline{A}_2 + T_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{13} = T_{11} \overline{A}_3 + T_{12} \overline{B}_w D_Q \overline{C}_e - \overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{14} = T_{11} \overline{A}_3 + T_{12} \overline{B}_w D_Q \overline{C}_e - \overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e,
\]

\[
t_{15} = T_{11} \overline{A}_3 + T_{12} \overline{B}_w D_Q \overline{C}_e - \overline{B}_w D_Q \overline{S}_1 + \overline{S}_1 \overline{B}_w D_Q - \overline{B}_w D_Q \overline{C}_Q - \overline{B}_w D_Q \overline{S}_{12} \overline{A}_1 + \overline{A}_1 \overline{S}_{12} \overline{A}_2 + \overline{A}_1 \overline{S}_{12} \overline{B}_w D_Q \overline{C}_e.
\]
Figures 5 shows the time responses obtained for a step reference of 0.5 for $y_1$, and 0.3 for $y_2$, and the disturbance rejection for a step disturbance of 0.05 applied to $u_1$ at $t = 2$ min. Figure 6 shows the control signals $u_1$ and $u_2$.

For robustness under additive uncertainties at high frequency, a high-pass filter is used for each control signal, as described in Section 4.2 which transfer form is $\tilde{W}_d = I_2(1 - 0.7q^{-1})/0.3$. Using the optimization procedure based on LMIs gives a multivariable Youla parameter as a $2 \times 2$ matrix of polynomials of order $n_f = 20$.

Figure 7 shows the singular values analysis of transfer from $b$ to control signals $u$ (from Figure 4). The greatest value of maximal singular values represents the $H_{\infty}$ norm. We can remark that this $H_{\infty}$ norm has been reduced. In this way the stability robustness is improved with respect to high-frequency additive unstructured uncertainties.

Figure 8 illustrates that the initial controller behaviour is destabilized by this uncertainty, but the robustified controller remains stable; it also shows the influence of the considered unstructured uncertainty to $y_2$.

6 CONCLUSIONS

This paper has presented a new MIMO complete methodology which enables robustifying an initial multivariable MPC controller in state-space formalism using the Youla parameter framework. In order to improve robustness towards unstructured uncertainties, a $H_{\infty}$ convex optimization problem was solved using the LMIs techniques. The major advantage of the developed structure is the state-space formulation of this MPC robustification problem for MIMO systems with a reduced computational effort compared to the transfer function formalism. This method can also be applied to non square systems, which otherwise are more difficult to control. This technique enables also the use of time-domain templates to manage the compromise between stability robustness and nominal performance.

REFERENCES