# SOLUTION OF THE FUNDAMENTAL LINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATION

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Abstract: This paper provides a solution of the fractional order system represented by the fundamental linear fractional order differential equation, namely, $(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t)$ whose transfer function is given by $G(s) = \frac{X(s)}{E(s)} = \frac{1}{[1+(\tau_0 s)^m]}$ for $0 < m < 2$. Simple methods of approximation, for a given frequency band, of the transfer function of this fractional order system by a rational function are presented. Analytical impulse and step responses of this system are derived. Illustrative examples are presented to show the exactitude of the approximation methods.

## 1 INTRODUCTION

In the recent decades the concepts of fractional order derivatives and integrals has been arisen in various areas of the engineering fields (Torvik,1984), (Ichise, 1971), (Sun, 1983), (Cole, 1941), (Davidson, 1950). Theses fractional concepts have been generally used to model physical systems, leading to the formulation of the linear fractional order differential equations. So, the dynamic systems described by this type of fractional differential equation are called fractional linear systems. With the growing number of applications system and control fields (Manabe, 1961), (Oustaloup, 1983), (Charef, 1992), (Podlubny, 1994), (Miller, 1993), (Hartley, 1998), (Petras, 2002), it is important to establish a clear system theory for these fractional order systems, so they may be accessible to the general engineering community.

The fundamental linear fractional order differential equation, defined in (Petras et al., 2002), is represented by the following equation:

$$(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t), \quad \text{for } 0 < m < 2 \qquad (1)$$

The transfer function of this type of fractional order systems is given by the following irrational function:

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[1+(\tau_0 s)^m]}, \quad \text{for} \quad 0 < m < 2 \qquad (2)$$

In this paper an effective and easy to use methods are presented for the approximation by a rational function, for a given frequency band, of the transfer function of the fundamental linear fractional order differential equation. Analytical impulse and step responses of this system are also derived. Illustrative examples are presented to show the exactitude and the usefulness of the approximation methods.

## 2 RELAXATION FRACTIONAL ORDER SYSTEM

### 2.1 Definition

Relaxation fractional order system is defined in this context as the fundamental linear fractional order differential equation of equation (1) with the transfer function of equation (2) for $0 < m < 1$.

### 2.2 Rational Function Approximation

In dielectric studies, Cole and Cole (Cole, 1941) observed that dispersion/relaxation data measured from a large number of materials can be modeled by the following function:

$$G(s) = \frac{1}{[1+(\tau_0 s)^m]}, \quad \text{for} \quad 0 < m < 1 \qquad (3)$$

It is also known that the distribution of relaxation times function $H(\tau)$ can be derived directly from the original transfer function as (MacDonald, 1987):

$$G(s) = \int_0^\infty \frac{H(\tau)}{1+s\tau} d\tau \qquad (4)$$

Cole and Cole (Cole, 1941) applied the above method to find the distribution of relaxation times function $H(\tau)$ for their model of equation (3) to be :

$$G(s) = \frac{1}{[1+(\tau_0 s)^m]} = \int_0^\infty \frac{H(\tau)}{1+s\tau} d\tau, \text{ for } \quad 0 < m < 1 \qquad (5)$$

with

$$H(\tau) = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log(\frac{\tau}{\tau_0})] - \cos[(1-m)\pi]}\right] \qquad (6)$$

The method of approximation began by sampling the distribution of relaxation times function $H(\tau)$ of equation (6) for a limited frequency band of approximation of practical interest $[0, \omega_H]$ at logarithmically equidistant points $\tau_i$ as follows (Sun, 1992):

$$H(\tau) \cong H_s(\tau) = \sum_{i=1}^{2N-1} H(\tau_i)\delta(\tau - \tau_i) \qquad (7)$$

and the points $\tau_i$ are such that:

$$\tau_i = \tau_0(\lambda)^{N-i} \quad \text{for } i = 1,2,\ldots,2N\text{-}1 \qquad (8)$$

with $\tau_N$ occurring at the characteristic relaxation time $\tau_0$, and $\lambda$, a constant positive real number greater than unity, is chosen such that:

$$\lambda = \frac{\tau_i}{\tau_{i+1}} \quad \text{for} \quad i = 1,2,\ldots,2N\text{-}1 \qquad (9)$$

Substituting equation (7) into equation (5), we obtain:

$$G(s) \cong \int_0^\infty \frac{\sum_{i=1}^{2N-1} H(\tau_i)\delta(\tau-\tau_i)}{1+s\tau} d\tau = \sum_{i=1}^{2N-1} \frac{H(\tau_i)}{1+s\tau_i} \qquad (10)$$

Hence, we can write that:

$$G(s) = \frac{1}{[1+(\tau_0 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)} \qquad (11)$$

where the $p_i$ 's are the poles of the approximation which are given as:

$$p_i = \frac{1}{\tau_i} = (\lambda)^{(i-N)} p_0 \quad, \text{for} \quad i = 1,2,\ldots,2N\text{-}1 \qquad (12)$$

such that $p_0 = 1/\tau_0$ and $\lambda = p_{i+1}/p_i$, the $k_i$ 's are the residues of the poles which are given from equation (6), for i = 1,2,...,2N-1, as:

$$k_i = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log(\frac{\tau_i}{\tau_0})] - \cos[(1-m)\pi]}\right] \qquad (13)$$

and for an approximation frequency $\omega_{max}$ which can be chosen to be $1000\omega_H$, with $[0, \omega_H]$ is the frequency band of practical interest, the number N is determined as follows:

$$N = \text{Integer}\left[\frac{\log(\tau_0\omega_{max})}{\log(\lambda)}\right] + 1 \qquad (14)$$

## 2.3 Time Responses

From equation (11), we have that:

$$G(s) = \frac{X(s)}{E(s)} = \frac{1}{[1+(\tau_0 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)} \qquad (15)$$

so,

$$X(s) = \frac{E(s)}{[1+(\tau_0 s)^m]} \cong \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)} E(s) \qquad (16)$$

for $e(t) = \delta(t)$ the unit impulse $E(s) = 1$, we will have

$$X(s) = \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)} \tag{17}$$

thus, the impulse response can be obtained as:

$$x(t) = \sum_{i=1}^{2N-1} k_i p_i \exp(-p_i t) \tag{18}$$

For e(t) = u(t) the unit step E(s) = 1/s, will be:

$$X(s) = \sum_{i=1}^{2N-1} \frac{k_i}{\left(1+\frac{s}{p_i}\right)} \frac{1}{s} = \sum_{i=1}^{2N-1} k_i \left(\frac{1}{s} - \frac{1}{s+p_i}\right) \tag{19}$$

thus, the step response can be obtained as:

$$x(t) = \sum_{i=1}^{2N-1} k_i \left(1-\exp(-p_i t)\right) \tag{20}$$

## 2.4 Illustrative Example

For illustration purpose let's take a numerical example for a relaxation fractional order system represented by the fundamental linear fractional order differential equation with m = 0.65 and $\tau_0 = 10$ as:

$$(10)^{0.65} \frac{d^{0.65}x(t)}{dt^{0.65}} + x(t) = e(t)$$

its transfer function is given by:

$$G(s) = \frac{1}{1+(10s)^{0.65}}$$

For a frequency band $[0, \omega_H]$ = [0, 100 rad/s], the approximation frequency $\omega_{max} = 1000\omega_H = 100000$ rad/s, $p_0 = 0.1$ rad/s and the ratio $\lambda = 4$, the number N, the poles $p_i$ and the residues $k_i$ of the approximation can be easily calculated from section (II.2) as: N=10, $p_i = (4)^{(i-N)} p_0$, for i = 1,2,...,19, and

$$k_i = \frac{1}{2\pi}\left[\frac{\sin[(1-m)\pi]}{\cosh[m\log((4)^{(10-i)})] - \cos[(1-m)\pi]}\right]$$

Figures (1) and (2) show the Bode plots of the relaxation fractional order system transfer function and its proposed rational function approximation. We can easily see that they are all quite overlapping over the frequency band of interest. Figures (3) and (4) show respectively the impulse and the step responses of this fractional order system obtained from its proposed rational function approximation.

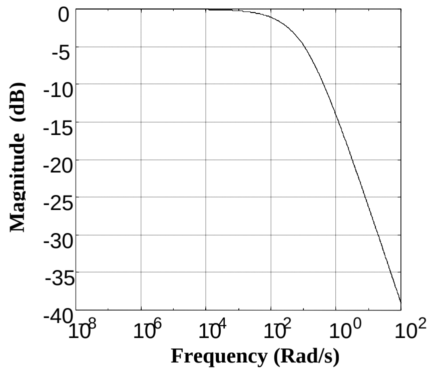


Figure 1: Magnitude of the Bode plot.

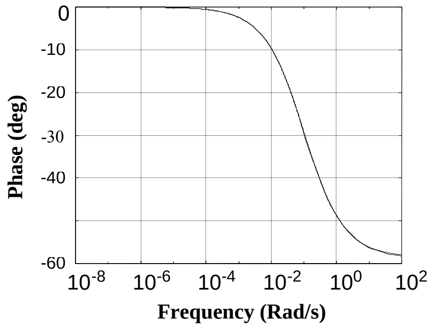


Figure 2: Phase of the Bode plot.

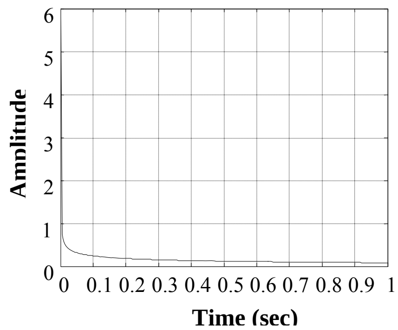


Figure 3: Impulse response.

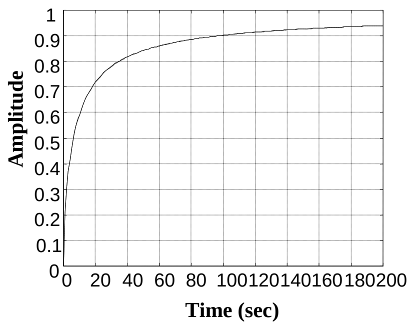


Figure 4: Step response.

# 3 OSCILLATION FRACTIONAL ORDER SYSTEM

## 3.1 Definition

Oscillation fractional order system is defined in this context as the fundamental linear fractional order differential equation of equation (1) with the transfer function of equation (2) for $1 < m < 2$.

## 3.2 Rational Function Approximation

First, the transfer function of the oscillation fractional order system is modeled as:

$$G(s)=\frac{1}{[1+(\tau_0 s)^m]}\cong\frac{(1+\tau_0 s)^{(2-m)}}{(\tau_0 s)^2+2\zeta(\tau_0 s)+1}=G_N(s)G_D(s) \quad (21)$$

$$G_N(s)=(1+\tau_0 s)^{(2-m)} \quad (22)$$

is a fractional power zero (FPZ) with $0 < (2-m) < 1$

$$G_D(s)=\frac{1}{(\tau_0 s)^2+2\zeta(\tau_0 s)+1} \quad (23)$$

is a regular second order system. It can be easily shown that:

for $\omega \ll 1/\tau_0$, $|G(j\omega)|=1\cong 1$

for $\omega \gg 1/\tau_0$, $|G(j\omega)|=\frac{1}{(\omega\tau_0)^m}\cong\frac{(\omega\tau_0)^{(2-m)}}{(\omega\tau_0)^2}=\frac{1}{(\omega\tau_0)^m}$

for $\omega = 1/\tau_0$, $|G(j\omega)|=\left|\frac{1}{(1+j^m)}\right|\cong\frac{|(1+j)^{(2-m)}|}{|j2\zeta|}$

$$|G(j\omega)|=\frac{1}{\sqrt{[(1+\cos(\frac{\pi}{2}m))^2+(\sin(\frac{\pi}{2}m))^2]}}\cong\frac{(\sqrt{2})^{2-m}}{2\zeta} \quad (24)$$

In order that the two sides of equation (24) were equal, the damping ratio $\zeta$ of the regular second order system must be given as:

$$\zeta=\sqrt{\frac{[1+\cos(\frac{\pi}{2}m)]}{2^{m-1}}} \quad (25)$$

To represent the oscillation fractional order system by a rational transfer function instead of the irrational function of equation (2), we have to approximate the FPZ of equation (22) by a rational one in a frequency band $[0, \omega_H]$. The method of approximation of the FPZ consists of approximating its 20(2-m) dB/dec slope on the Bode plot by a number of zig-zag lines with alternate slopes of 20 dB/dec and 0 dB/dec corresponding to alternate zeros and poles on the negative real axis of the s-plane such that $z_0 < p_0 < z_1 < p_1 < \ldots < z_N < p_N$. Hence, we can write that:

$$G_N(s)=(1+\tau_0 s)^{(2-m)}\cong\frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)} \quad (26)$$

So, equation (21) can be rewritten as:

$$G(s)=\frac{1}{[1+(\tau_0 s)^m]}\cong\frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)}\frac{1}{\left[(\tau_0 s)^2+2\zeta(\tau_0 s)+1\right]} \quad (27)$$

As the same idea of the method used to approximate the fractional power pole (Charef, 1992), the approximation of the ZPF began with a specified approximation error y in dB and an approximation frequency band $\omega_{max}$ which can be $100\omega_H$, then the parameters a, b, $z_0$, $p_0$ and N of the approximation can be easily determined as follows:

$$a=10^{\left[\frac{y}{10[1-(2-m)]}\right]},\ b=10^{\left[\frac{y}{10(2-m)}\right]},\ z_0=\frac{1}{\tau_0}10^{\left[\frac{y}{20(2-m)}\right]}$$

$$p_0=az_0, \text{ and } N=\text{Integer}\left[\frac{\log\left(\frac{\omega_{max}}{z_0}\right)}{\log(ab)}\right]+1$$

Hence, the zeros $z_i$'s and the poles $p_i$'s of equation (27) can then be derived from the above parameters for i=0,1,…,N as: $z_i=z_0(ab)^i$ and $p_i=p_0(ab)^i$. Then, equation (27) can be rewritten as:

$$G(s)=\frac{1}{[1+(\tau_0 s)^m]}=\frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_0(ab)^i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_0(ab)^i}\right)}\frac{1}{\left[(\tau_0 s)^2+2\zeta(\tau_0 s)+1\right]} \quad (28)$$

## 3.3 Time Responses

By partial fraction expansion of the rational function of equation (28) it is possible to represent the transfer function of the oscillation fractional order system by a linear combination of elementary simple functions, that is:

$$G(s) = \sum_{i=0}^{N} \frac{k_i}{\left(1+\frac{s}{p_0(ab)^i}\right)} + \frac{As+B}{(\tau_0 s)^2 + 2\zeta(\tau_0 s)+1} \quad (29)$$

where the $k_i$ (i=0,1, …, N) are the residues of the poles which can be calculated as:

$$k_i = \frac{\prod_{j=0}^{N}\left[1-a(ab)^{(i-j)}\right]}{\prod_{\substack{j=0\\ i\neq j}}^{N}\left[1-(ab)^{(i-j)}\right]}\left\{\frac{1}{\left(\tau_0 p_0(ab)^i\right)^2 - 2\zeta\left(\tau_0 p_0(ab)^i\right)+1}\right\} \quad (30)$$

and the constants A and B can also be calculated as:

at s = 0, $G(0) = B + \sum_{i=0}^{N} k_i = 1$, then $B = 1 - \sum_{i=0}^{N} k_i$, also

$\lim_{s\to\infty} sG(s) = 0 = \frac{A}{\tau_0^2} + \sum_{i=0}^{N} k_i p_0 (ab)^i$, then $A = -\tau_0^2 \sum_{i=0}^{N} k_i p_0 (ab)^i$

We will then have that:

$$G(s) = \frac{X(s)}{E(s)} = \sum_{i=0}^{N} \frac{k_i}{\left(1+\frac{s}{p_0(ab)^i}\right)} + \frac{As+B}{(\tau_0 s)^2 + 2\zeta(\tau_0 s)+1} \quad (31)$$

$$X(s) = \sum_{i=0}^{N} \frac{k_i}{\left(1+\frac{s}{p_0(ab)^i}\right)} E(s) + \frac{As+B}{(\tau_0 s)^2 + 2\zeta(\tau_0 s)+1} E(s) \quad (32)$$

for e(t) = δ(t) the unit impulse E(s) = 1, the impulse response of this system is given as:

$$x(t) = \sum_{i=0}^{N} k_i p_0 (ab)^i \exp\left(-p_0(ab)^i t\right) + C\exp\left(-\frac{\zeta}{\tau_0}t\right)\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau_0}t + \Phi\right) \quad (33)$$

where the constants C and Φ are given as (17):

$$C = \frac{B}{\tau_0}\sqrt{\frac{A^2 - 2AB\zeta\tau_0 + (B\tau_0)^2}{(B\tau_0)^2\left(1-\zeta^2\right)}}$$

$$\Phi = \operatorname{arctg}\left(\frac{A\sqrt{1-\zeta^2}}{B\tau_0 - A\zeta}\right)$$

Now, for e(t) = u(t) the unit step E(s) = 1/s, equation (32) we will be

$$X(s) = \sum_{i=0}^{N} \frac{k_i}{\left(1+\frac{s}{p_0(ab)^i}\right)}\frac{1}{s} + \frac{As+B}{(\tau_0 s)^2 + 2\zeta(\tau_0 s)+1}\frac{1}{s} \quad (34)$$

the step response of this system can be obtained as:

$$x(t) = 1 - \sum_{i=0}^{N} k_i \exp\left(-p_0(ab)^i t\right) + C_1\exp\left(-\frac{\zeta}{\tau_0}t\right)\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau_0}t + \Phi_1\right) \quad (35)$$

where the constants $C_1$ and $\Phi_1$ are given as (Kuo, 1987):

$$C_1 = B\sqrt{\frac{A^2 - 2AB\zeta\tau_0 + (B\tau_0)^2}{(B\tau_0)^2\left(1-\zeta^2\right)}}$$

$$\Phi_1 = \operatorname{arctg}\left(\frac{A\sqrt{1-\zeta^2}}{B\tau_0 - A\zeta}\right) - \operatorname{arctg}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right)$$

## 3.4 Illustrative Example

Let's take a numerical example for an oscillation fractional order system represented by the following fundamental linear fractional order differential equation with m = 1.7 and $\tau_0$ = 0.1 as:

$$(0.1)^{1.7}\frac{d^{1.7}x(t)}{dt^{1.7}} + x(t) = e(t)$$

its transfer function is given by:

$$G(s) = \frac{1}{1+(0.1s)^{1.7}}$$

First, G(s) is modeled by the following function:

$$G(s) = \frac{1}{[1+(0.1s)^{1.7}]} = \frac{(1+0.1s)^{(0.3)}}{(0.1s)^2 + 0.52(0.1s)+1}$$

For a frequency band of practical interest [0, $\omega_H$] = [0, 1000 rad/s], the approximation of the fractional power zero $(1+0.1s)^{(0.3)}$ by a rational function is given as:

$$(1+0.1s)^{(0.3)} = \frac{\prod_{i=0}^{N}\left(1+\frac{s}{z_0(ab)^i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_0(ab)^i}\right)}$$

for an approximation error y = 1 dB and an approximation frequency band $\omega_{max}$ =100$\omega_H$ = 100000 rad/s, the parameters a, b, $z_0$, $p_0$ and N of the above equation can be easily calculated as follows : a = 1.389, b = 2.154, $z_0$ = 14.678 rad/s, $p_0$ = 20.395 rad/s and N = 9, so:

$$(1+0.1s)^{(0.3)} = \frac{\prod_{i=0}^{9}\left(1+\frac{s}{14.678(2.993)^i}\right)}{\prod_{i=0}^{9}\left(1+\frac{s}{20.395(2.993)^i}\right)}$$

then, we will have that:

$$G(s) = \frac{\prod_{i=0}^{9}\left(1+\frac{s}{14.678(2.993)^i}\right)}{\prod_{i=0}^{9}\left(1+\frac{s}{20.395(2.993)^i}\right)} \frac{1}{(0.1s)^2 + 0.52(0.1s) + 1}$$

Figures (5) and (6) show the Bode plots of the system transfer function and its proposed rational function approximation. Figures (7) and (8) show respectively the impulse and the step responses of the system obtained from its proposed rational function approximation.

## 4 CONCLUSION

In this paper I have presented some effective methods for approximating the irrational function given by $G(s) = \frac{1}{[1+(\tau_0 s)^m]}$, for 0 < m < 2, representing the transfer function of the fundamental linear fractional order differential equation $(\tau_0)^m \frac{d^m x(t)}{dt^m} + x(t) = e(t)$ by a rational function, in a given frequency band. The impulse and step responses of this type of systems are derived. Illustrative examples have been treated to demonstrate the usefulness of the approximation methods.

Theses approximations can very suitable for analysis, realization and implementation of fractional order systems. The expressions for characteristics and usual time and frequency specifications can also be derived.

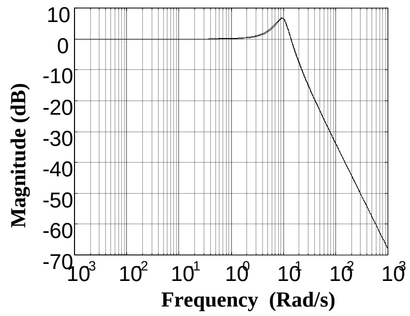


Figure 5: Magnitude Bode plot.

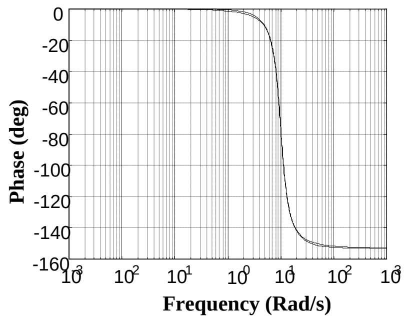


Figure 6: Phase of the Bode plot.

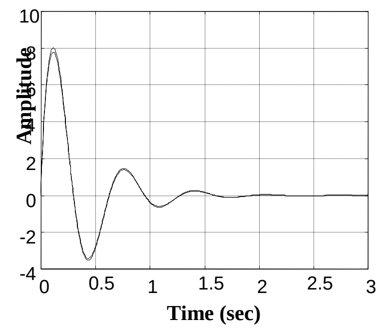


Figure 7: Impulse response.

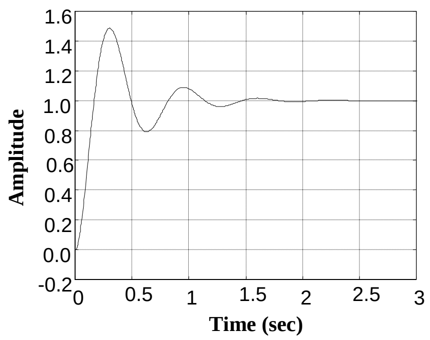


Figure 8: Step response.

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