CRONE OBSERVER
Definition and Design Methodology

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Abstract: CRONE control, robust control methodology based on fractional differentiation, is applied to state observer design. State observation can indeed be viewed as a regulation problem given that the goal of a state observer is to cancel the observation errors in spite of measurement noises, disturbances and plant perturbations. This conclusion has been used recently to define a new class of state observers known in the literature as "dynamic observers" or "input-output observer". It is based on the observation error dynamic feedback. In this paper, this idea is used to define the CRONE observer design methodology. Performance robustness of the obtained observers versus plant perturbations is analysed. As for CRONE control, fractional differentiation in the definition of an equivalent open loop transfer function permits to reduce the number of parameters to be optimised.

1 INTRODUCTION

In many industrial applications of control, controlled variables cannot be directly measured by sensors. In such a situation, these variables can be reconstructed with a Luenberger type observer (Luenberger, 1971). However, it is really difficult to take into account modelling errors and disturbances in the synthesis the observer gains. We recently faced with this problem, for the speed control of a steel rolling mill, speed of the load being not measured due the high temperatures and maintenance costs (Sabatier et *al*., 2003). Moreover, some parameters of the system were not known with accuracy (such as sliding viscous coefficients). To solve this problem, a Luenberger observer was associated with a CRONE controller (Oustaloup, 1991). In this application of CRONE control, an overestimation of the plant uncertainties was required to take into account bias introduced by the observer due to differences between plant and observer model behaviours as the time of plant parameters variations. To reduce the resulting conservatism, a robust observer has to be designed, robustness of the observation error convergence to zero in spite of disturbances and plant perturbation being addressed.
A solution to obtain such an observer, consists in considering observation problem as a classic regulation problem and thus to construct a feedback loop with the available information (plant input and output), whose goal is to cancel the observation errors in spite of measurement noise, disturbances and plant perturbations. This new concept was recently published and applied on a real system (Marquez, 2003) (Marquez and Riaz, 2005). In this paper, a CRONE controller is introduced in the feedback loop in order to take into account the disturbances and the model perturbation. In comparison with the H_∞ approach used by Marquez, plant model perturbations are taken into account in a structured form with no overestimation, thus, without conservatism. Due to the introduction of fractional differentiation in the CRONE approach, an open loop transfer function with only three parameters (just like a PID controller) has to be optimised to simultaneously reduce the effects of disturbances and model perturbation on the observation error. Another contribution of the paper is the extension of the idea by Marquez to the problem of state observation with unknown input.
The paper is organised as follows. Section 2 presents the dynamic output feedback based observer concept developed in (Marquez, 2003) (Marquez and Riaz, 2005) and extends it to observation with unknown input. Section 3 gives some generalities on CRONE control. In section 4, application of CRONE control

to state observation problem is developed thus defining an observer that will be referred to as a CRONE observer in future developments.

2 DYNAMIC OUTPUT FEEDBACK BASED OBSERVER

2.1 Presentation

Dynamic output feedback based observer concept was introduced in (Marquez, 2003) and (Marquez and Riaz, 2005) in which the observation problem is solved using the feedback diagram of Fig. 1. The plant P, the model M and the dynamic controller K are supposed single input / single output systems represented by the state space descriptions:

$$P:\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{cases}\qquad(1)$$

$$M:\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+Bv(t)=A\hat{x}(t)+B(u(t)+w(t))\\ \hat{y}(t)=C\hat{x}(t)\end{cases}\qquad(2)$$

$$K:\begin{cases}\dot{x}_K(t)=A_k x_K(t)-B_K\varepsilon(t)\\ \qquad\quad =A_K x(t)+B_K C(x(t)-\hat{x}(t))\\ w(t)=C_K x_K(t)\end{cases}\qquad(3)$$

State $x(t)$ is supposed not measurable and $\hat{x}(t)$ denotes the estimated state. All the elements of matrices and vectors in (1) to (3) are supposed element of $\mathbb{R}$.

Figure 1 clearly shows that the goal of the used feedback structure is to cancel the observation error $\chi(t)=x(t)-\hat{x}(t)$ by cancelling the error signal $\varepsilon=\hat{y}(t)-y(t)$. Time derivative of the observation error $\dot{\chi}(t)=\dot{x}(t)-\dot{\hat{x}}(t)$, is thus given by :

$$\begin{aligned}\dot{\chi}(t)&=Ax(t)+Bu(t)-Ax(t)-B(u(t)+w(t))\\ &=A\chi(t)-BC_K x_K(t)\end{aligned}\quad.\qquad(4)$$

Using controller state space description (3), a state space description for the system in Fig. 1 involving the observation error is thus:

$$\begin{bmatrix}\dot{\chi}(t)\\ \dot{x}_K(t)\end{bmatrix}=\begin{bmatrix}A & -BC_K\\ B_K C & A_K\end{bmatrix}\begin{bmatrix}\chi(t)\\ x_K(t)\end{bmatrix}=A_O\begin{bmatrix}\chi(t)\\ x_K(t)\end{bmatrix}\qquad(5)$$

Matrix A_O in relation (5) is also the state matrix of the feedback system in Fig. 2. Such a remark permits to demonstrate the following theorem.

Theorem (Marquez, 2003)
State $\hat{x}(t)$ exponentially converge to the state $x(t)$ with the feedback structure of Fig. 1, if all matrix A_O eigenvalues of has a strictly negative part or if the system in Fig. 2 is internally stable.

2.2 Extension to State Observation with Unknown Input

The problem of state observation with unknown input is now addressed using the dynamic output feedback structure of Fig. 3.
It is supposed that the plant P and the model M are described by the following state space descriptions:

$$P:\begin{cases}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{cases}\qquad(6)$$

$$M:\begin{cases}\dot{z}(t)=Nz(t)+Bv(t)\\ \hat{x}(t)=z(t)-Ey(t)\\ \hat{y}(t)=C\hat{x}(t)\end{cases}\qquad(7)$$

Observation error dynamics is thus defined by:

$$\dot{\chi}(t)=\dot{x}(t)-\dot{\hat{x}}(t)=Ax(t)+Bu(t)-(\dot{z}(t)-E\dot{y}(t))\qquad(8)$$

or using relations (6) and (7):

$$\begin{aligned}\dot{\chi}(t)=&Ax(t)-NECx(t)+ECAx(t)\\ &-N\hat{x}(t)+Bu(t)+ECBu(t)-Bv(t)\end{aligned}\quad.\qquad(9)$$

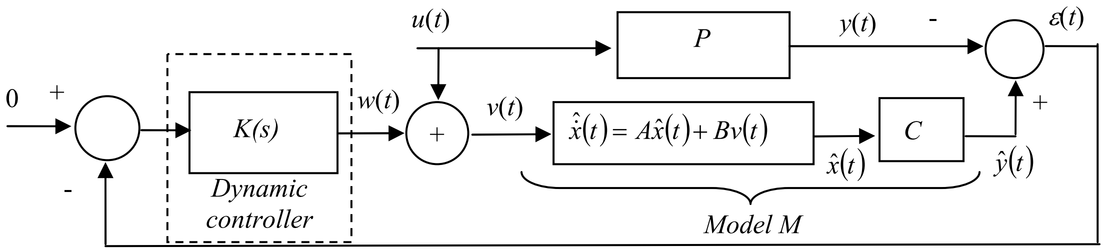

Figure 1: Dynamic output feedback based observer.

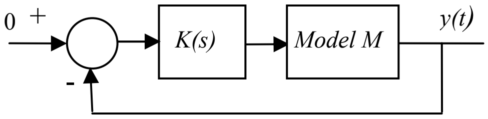

Figure 2: Feedback system with state matrix A_O.

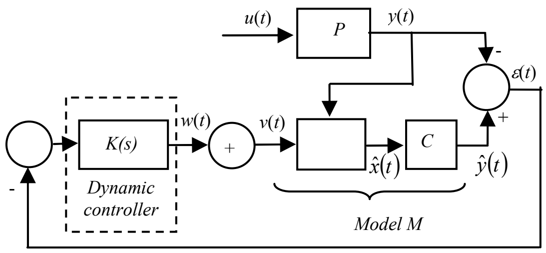

Figure 3: Dynamic output feedback based observer with unknown input.

Suppose now that matrix E is such that

$$B + ECB = 0 \qquad \text{or} \qquad E = -B(CB)^* \tag{10}$$

in which $(CB)^*$ denotes the generalised inverse of CB if it exits. Equation (9) thus becomes:

$$\dot{\chi}(t) = Ax(t) - NECx(t) + ECAx(t) - N\hat{x}(t) - Bv(t) \tag{11}$$

or using the state space description of the controller K of relation (3):

$$\begin{cases} \dot{\chi}(t) = Ax(t) - NECx(t) + ECAx(t) \\ \qquad\qquad - N\hat{x}(t) - BC_K x_K(t) \\ \dot{x}_K(t) = A_K x_K(t) + B_K C\chi \end{cases}. \tag{12}$$

Let now

$$P = I + EC \qquad \text{and thus} \qquad EC = P - I \tag{13}$$

then

$$\begin{aligned} A - NEC + ECA &= A - N(P-I) + (P-I)A \\ &= -NP + N + PA \end{aligned}. \tag{14}$$

If it is now imposed now that

$$-NP + PA = 0 \quad \text{and thus} \quad N = PAP^{-1}, \tag{15}$$

equation (12) becomes:

$$\begin{bmatrix} \dot{\chi}(t) \\ \dot{x}_K(t) \end{bmatrix} = \begin{bmatrix} N & -BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} \chi(t) \\ x_K(t) \end{bmatrix}. \tag{16}$$

Relation (16) is similar to relation (5) and thus highlights, given the analysis following relation (5), that the observation error converges exponentially to zero if controller K internally stabilise the feedback system in Fig. 2, model M being defined by:

$$M : \begin{cases} \dot{\chi}(t) = N\chi(t) + Bv(t) \\ y(t) = C\chi(t) \end{cases}. \tag{17}$$

3 CRONE CSD PRINCIPLES

3.1 Introduction to Fractional Integro-Differentiation

The first definitions of fractional order differentiation (or integration) were given by Leibniz and Euler at the end of the 17th and during the 18th century. In the 19th century many mathematicians generalized these definitions: Laplace, Lacroix, Fourier, Liouville, Abel, Hargreave, Riemann etc. In 1869 Sonin extended the Cauchy integral to fractional integration orders and the Riemann-Liouville definition was finally proposed.

Operational calculus can also be used. Let $y(t)$ be the order n derivative of the causal signal $x(t)$:

$$y(t) = x^{\{n\}}(t) = D^n x(t) \tag{18}$$

with $n \in \mathbb{C}$ and where D is the differentiation operator. If the real part of n is negative, then $y(t)$ is in fact the order $-n$ integral of $x(t)$.

The transfer function of the linear operator D^n is defined by the Laplace transform:

$$D(s) = L\{y(t)\} / L\{x(t)\} = s^n. \tag{19}$$

Its impulse response is given by:

$$d(t) = L^{-1}\{s^n\} = \frac{t^{-n-1}}{\Gamma(-n)} \mathrm{H}(t) \text{ if } \mathrm{Re}[n] \in \mathbb{R}^- \text{ and } \mathrm{Re}[n] \neq 0,$$

$$\text{or } \forall t \neq 0 \text{ if } \mathrm{Re}[n] \in \mathbb{R}^+ - \mathbb{N} \text{ and } \mathrm{Re}[n] = 0 \tag{20}$$

where $\Gamma(.)$ and $H(.)$ denote the gamma and Heaviside functions.

Convoluting $d(t)$ and $x(t)$, $y(t)$ can be computed using the following integrals:

$$y(t) = \int_0^t \frac{\theta^{-n-1}}{\Gamma(-n)} x(t-\theta) d\theta \tag{21}$$

if $\mathrm{Re}[n] \in \mathbb{R}^-$ and $\mathrm{Re}[n] \neq 0$ which is the Riemann-Liouville definition, and

$$y(t) = \left(\frac{d}{dt}\right)^m \int_0^t \frac{\theta^{m-n-1}}{\Gamma(-n)} x(t-\theta) d\theta \tag{22}$$

if $\mathrm{Re}[n] \in \mathbb{R}^+ - \mathbb{N}$ and $\mathrm{Re}[n]=0$, where m is defined by the integer part of the real part of n. It is obvious that a specificity of this fractional differentiation, is that it takes into account all the past of signal $x(t)$. A fractional-order system can be considered as an infinite order rational system. Thus, fractional systems are often used to model distributed parameter systems. As fractional operators can replace high order transfer functions in system-identification or control-system design, they are also used to determine models or controllers with few tuning parameters.

Since the sixties, some electrical circuits have been proposed for synthesizing half order differentiators (Suezaki and Takahashi,1966), (Dutta Roy, 1970), (Biorci and Ridella, 1970), (Ichise *et al.*, 1971), (Oldham, 1973). From 1975 on, Oustaloup et al. proposed methodologies for synthesizing band-limited differentiators whose orders are fractional (Oustaloup, 1975). Since 1990, they have extended this to complex fractional order differentiators (Oustaloup et al., 1990), (Oustaloup et al., 2000) and have applied it to robust control design. Fractional or non-integer order systems are also termed Warburg impedance or Constant Phase Element (CPE), and are associated to long-time memory behaviours.

3.2 Introduction to the CRONE Methodology

The CRONE control-system design is based on the common unity-feedback configuration (Fig. 4). The robust controller or the open-loop transfer function is defined using fractional order integro-differentiation. The required robustness is that of both stability margins and performance, and particularly peak value M_r (called resonant peak) of the common complementary sensitivity function $T(s)$.

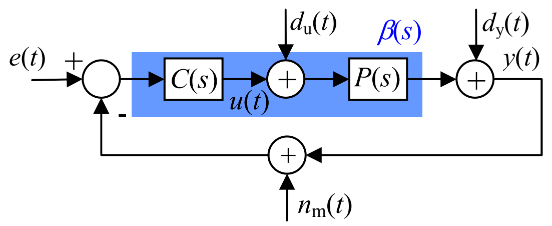

Figure 4: Common CRONE control-system diagram.

Three CRONE control design methods have been developed, successively extending the application field.

The third CRONE control generation must be used when the plant frequency uncertainty domains are of various types (not only gain-like). It is based on the definition of a generalized template described as a straight line in the Nichols chart of any direction (complex fractional order integration), or by a multi-template (or curvilinear template) defined by a set of generalized templates.

An optimization allows the determination of the independent parameters of the open loop transfer function. This optimization is based on the minimization of the stability degree variations, while respecting other specifications taken into account by constraints on sensitivity function magnitude. The complex fractional order permits parameterization of the open-loop transfer function with a small number of high-level parameters. The optimization of the control is thus reduced to only the search for the optimal values of these parameters. As the form of uncertainties taken into account is structured, this optimization is necessarily nonlinear. It is thus very important to limit the number of parameters to be optimized. After this optimization, the corresponding CRONE controller is synthesized as a rational fraction only for the optimal open-loop transfer function.

The third generation CRONE CSD methodology, the most powerful one, is able to design controllers for plants with positive real part zeros or poles, time delay, and/or with lightly damped modes (Oustaloup *et al.*,1995). Associated with the w-bilinear variable change, it also permits the design of digital controllers. The CRONE control has also been extended to linear time variant systems and nonlinear systems whose nonlinear behaviors are taken into account by sets of linear equivalent behaviors (Pommier *et al.*, 2002). For MIMO (multivariable) plants, two methods have been developed (Lanusse et *al.*, 2000). The choice of the method is made through an analysis of the coupling rate of the plant. When this rate is reasonable, one can opt for the simplicity of the multi SISO approach.

3.3 Third Generation CRONE Methodology

Within a frequency range $[\omega_A, \omega_B]$ around open-loop gain-crossover frequency ω_{cg}, the Nichols locus of a third generation CRONE open-loop is defined by an any-angle straight line segment, called a generalized template (Fig. 5).

The generalized template can be defined by an integrator of complex fractional order n whose real part determines its phase location at frequency ω_{cg}, that is $-\mathrm{Re}_{/i}(n)\pi/2$, and whose imaginary part then determines its angle to the vertical (Fig. 5).

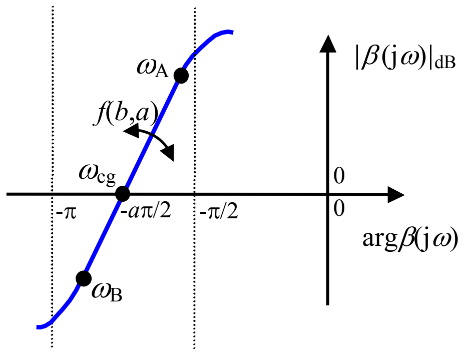

Figure 5: Generalized template in the Nichols plane.

The transfer function including complex fractional order integration is:

$$\beta(s)=\left(\cosh\left(b\frac{\pi}{2}\right)\right)^{\mathrm{sign}(b)}\left(\frac{\omega_{cg}}{s}\right)^{a}\left(\mathrm{Re}_{/i}\left(\left(\frac{\omega_{cg}}{s}\right)^{ib}\right)\right)^{-\mathrm{sign}(b)} \tag{23}$$

with $n = a + ib \in \mathbb{C}_i$ and $\omega \in \mathbb{C}_j$, and where $\mathbb{C}_i$ and $\mathbb{C}_j$ are respectively time-domain and frequency-domain complex planes. In (Hartley and Lorenzo, 2005) a physical interpretation of such a complex order operator is proposed.

The definition of the open-loop transfer function including the nominal plant must take into account:

- accuracy specifications at low frequencies;
- the generalized template around frequency ω_{cg};
- plant behaviour at high frequencies while respecting the control effort specifications at these frequencies.

Thus, the open-loop transfer function is defined by a transfer function using band-limited complex fractional order integration:

$$\beta(s)=\beta_l(s)\bar{\beta}(s)\beta_h(s), \tag{24}$$

with:

$$\bar{\beta}(s)=C^{\mathrm{sign}(b)}\left(\alpha\frac{1+s/\omega_h}{1+s/\omega_l}\right)^{a}\left(\Re e_{/i}\left\{\left(\alpha\frac{1+s/\omega_h}{1+s/\omega_l}\right)^{ib}\right\}\right)^{-q\,\mathrm{sign}(b)} \tag{25}$$

$$\alpha_0=\left(1+\left(\frac{\omega_r}{\omega_0}\right)^2\Big/1+\left(\frac{\omega_r}{\omega_1}\right)^2\right)^{1/2} \tag{26}$$

where $\beta_l(s)$ is an integer order n_l proportional integrator:

$$\beta_l(s)=C_l\left(\frac{\omega_l}{s}+1\right)^{n_l} \tag{27}$$

- where $\beta_h(s)$ is a low-pass filter of integer order n_h:

$$\beta_h(s)=\frac{C_h}{\left(\dfrac{s}{\omega_h}+1\right)^{n_h}} \tag{28}$$

with

$$C_l=\left(\frac{\omega_{cg}^{\,2}}{\omega_l^{\,2}+\omega_{cg}^{\,2}}\right)^{n_l/2} \quad\text{and}\quad C_h=\left(\frac{\omega_{cg}^{\,2}}{\omega_h^{\,2}}+1\right)^{n_h/2} \tag{29}$$

The optimal open loop transfer function is obtained by the minimization of the robustness cost function

$$J=\sup_{\omega,P}\left|T(j\omega)\right|-M_{r0}\ , \tag{30}$$

where M_{r0} is the resonant peak set for the nominal parametric state of the plant, while respecting the following set of inequality constraints for all plants (or parametric states of the plant) and for $\omega \in \mathbb{R}^+$:

$$\inf_P\left|T(j\omega)\right|\geq T_l(\omega) \ \text{and}\ \sup_P\left|T(j\omega)\right|\leq T_u(\omega)\ , \tag{31}$$

$$\sup_P\left|S(j\omega)\right|\leq S_u(\omega)\ ,\ \sup_P\left|CS(j\omega)\right|\leq CS_u(\omega)$$

and

$$\sup_P\left|PS(j\omega)\right|\leq PS_u(\omega)\ , \tag{32}$$

with

$$\begin{cases} T(s)=\dfrac{C(s)P(s)}{1+C(s)P(s)} & S(s)=\dfrac{1}{1+C(s)P(s)} \\ CS(s)=\dfrac{C(s)}{1+C(s)P(s)} & PS(s)=\dfrac{G(s)}{1+C(s)P(s)} \end{cases} \tag{33}$$

As the uncertainties are taken into account by the least conservative method, a non-linear optimization method must be used to find the optimal values of three independent parameters. The parameterization of the open-loop transfer function by complex fractional orders, then simplifies the optimization considerably. During optimization a complex order has the same function as a whole set of parameters found in common rational controllers.

When the optimal nominal open-loop transfer is determined, the fractional controller $K_F(s)$ is defined by its frequency response:

$$K_F(j\omega)=\frac{\beta(j\omega)}{P_0(j\omega)}, \tag{34}$$

where $P_0(j\omega)$ is the nominal frequency response of the plant.

The parameters of a rational transfer function $K_R(s)$ with a predefined low-order structure are tuned to fit the ideal frequency response $K_F(j\omega)$. The rational integer model on which the parametric estimation is based, is given by:

$$K_{\mathrm{R}}(s) = \frac{B(s)}{A(s)}\ , \tag{35}$$

where $B(s)$ and $A(s)$ are polynomials of specified integer degrees n_B and n_A. Any frequency-domain system-identification technique can be used. An advantage of this design method is that whatever the complexity of the control problem, satisfactorily low values of n_B and n_A, usually around 6, can be used without performance reduction.

4 CRONE OBSERVER

Robustness considerations versus plant perturbation are also addressed in (Marquez, 2003) in an H_∞ framework for the synthesis of an dynamic output feedback based observer. In this paper, robustness to plant perturbation is taken into account with CRONE Control, thus leading to a new formulation of in the CRONE control-system design methodology.

4.1 Plant Perturbations and Disturbance Rejection Effects

It is now supposed that the plant whose state is estimated is submitted to perturbations. Effects of these perturbations but also effects of output disturbances $d_y(t)$ and measurement noises $n(t)$ on the estimation error are now studied. Control diagram of Fig. 6 is considered.
Using the notations previously introduced for the plant P, the model M and the controller K, the following state space description are now manipulated:

$$P: \begin{cases} \dot{x}(t) = (A+\Delta_A)x(t) + (B+\Delta_B)(u(t)+d_u(t)) \\ y(t) = Cx(t) + d_y(t) + n(t) \end{cases} \tag{36}$$

$$M: \begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bv(t) = A\hat{x}(t) + B(u(t)+w(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{37}$$

$$K: \begin{cases} \dot{x}_K(t) = A_k x_K(t) - B_K\varepsilon(t) \\ \qquad = A_K x(t) + B_K(Cx(t) + d_y(t) + n(t)) - B_K C\hat{x}(t) \\ w(t) = C_K x_K(t) \end{cases} \tag{38}$$

Δ_A and Δ_B are real matrices of appropriate dimensions that models plant perturbations.
At time $t = 0$, it is supposed that $x_K(0) = 0$, $x(0) = x_0$, $\hat{x}(0) = 0$ and thus $\chi(0) = x_0 = \chi_0$.
Laplace transform applied to relations (36) to (38) thus lead to:

$$P: \begin{cases} [sI - (A+\Delta_A)]x(s) = \chi_0 + (B+\Delta_B)u(s) \\ y(s) = Cx(s) + d_y(s) + n(s) \end{cases} \tag{39}$$

$$M: \begin{cases} [sI - A]\hat{x}(s) = B(u(s)+w(s)) \\ \hat{y}(s) = C\hat{x}(s) \end{cases} \tag{40}$$

and

$$K: \begin{cases} x_K(s) = [sI - A_K]^{-1}\begin{pmatrix} B_K(Cx(s) + d_y(s) + n(s)) \\ -B_K C\hat{x}(s) \end{pmatrix} \\ w(s) = C_K x_K(s) \end{cases} . \tag{41}$$

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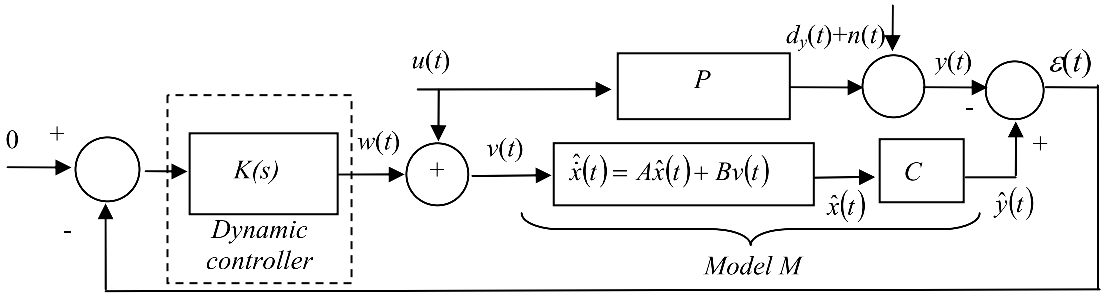

Figure 6: Dynamic output feedback based observer.

$$K:\begin{cases} x_K(s)=[sI-A_K]^{-1}\begin{pmatrix} B_K\left(Cx(s)+d_y(s)+n(s)\right) \\ -B_K C\hat{x}(s)\end{pmatrix} \\ w(s)=C_K x_K(s)\end{cases}. \quad (41)$$

Difference of state equations of representations (39) and (40) gives:

$$\begin{aligned}&[sI-(A+\Delta_A)]x(s)-[sI-A]\hat{x}(s)\\ &\quad=\chi_0+(B+\Delta_B)u(s)-B(u(s)+w(s))\end{aligned} \quad (42)$$

and thus using output equation of representation (41):

$$\begin{aligned}&[sI-(A+\Delta_A)]x(s)-[sI-A]\hat{x}(s)\\ &\quad=\chi_0+(B+\Delta_B)u(s)-B(u(s)+C_K x_K(s))\end{aligned} \quad (43)$$

Let $K(s)$ denotes the transfer function of the controller K, with:

$$K(s)=C_K[sI-A_K]^{-1}B_K, \quad (44)$$

Then relation (43) becomes:

$$\begin{aligned}&[sI-(A+\Delta_A)]x(s)-[sI-A]\hat{x}(s)\\ &\quad=\chi_0+(B+\Delta_B)u(s)-Bu(s)\\ &\qquad-BC_K[sI-A_K]^{-1}B_K C(x(s)-\hat{x}(s))\\ &\qquad-BC_K[sI-A_K]^{-1}B_K\left(d_y(s)+n(s)\right)\end{aligned} \quad (45)$$

and thus

$$\begin{aligned}&[sI-(A+\Delta_A)]x(s)-[sI-A]\hat{x}(s)\\ &\quad=\chi_0+(B+\Delta_B)u(s)-Bu(s)\\ &\qquad-BK(s)C(x(s)-\hat{x}(s))-BK(s)\left(d_y(s)+n(s)\right)\end{aligned}. \quad (46)$$

Laplace transform of the observation error is thus given by:

$$\begin{aligned}\chi(s)&=[sI-A+BK(s)C]^{-1}\chi_0\\ &\quad+[sI-A+BK(s)C]^{-1}\Delta_A x(s)\\ &\quad+[sI-A+BK(s)C]^{-1}\Delta_B u(s)\\ &\quad-[sI-A+BK(s)C]^{-1}BK(s)\left(d_y(s)+n(s)\right)\end{aligned} \quad (47)$$

4.2 Crone Observer Synthesis

Relation (47) demonstrates that without disturbances and plant perturbations ($\Delta_A=0$, $\Delta_B=0$, $d_y(s)+n(s)=0$) observation error converges exponentially to 0 if the roots of the determinant of transfer matrix $[sI-A+BK(s)C]^{-1}$ lie in the left half complex plane, or equivalently given comments before theorem 1, if the closed loop in Fig. 2 is internally stable.

Moreover, relation (47) demonstrates that with disturbances and plant perturbations, observation errors can be reduced by finding a controller $K(s)$ that minimizes the modulus of the elements of the transfer matrix $[sI-A+BK(s)C]^{-1}\Delta_A$ and vectors $[sI-A+BK(s)C]^{-1}\Delta_B$ and $[sI-A+BK(s)C]^{-1}BK(s)$. Also notes that final value theorem can be applied on the elements on the previous matrix and vectors, to analyse the effects of plant perturbation and disturbances on observation error.

CRONE observer synthesis thus consist in finding an optimal open loop behaviour defined by transmittance (25) that minimises the maximal gain of matrix $[j\omega I-A+BK(j\omega)C]^{-1}\Delta_A$ and vectors $[j\omega I-A+BK(j\omega)C]^{-1}\Delta_B$ and $[j\omega I-A+BK(j\omega)C]^{-1}BK(j\omega)$ as ω varies within the frequency range $]0,..,\infty[$.

An algorithm for the CRONE observer synthesis can thus be summarized as follows:

- choice of an open-loop gain-crossover frequency ω_{cg} that ensures a satisfactory observation error cancellation dynamics;
- choice of orders n_l and n_h in order to ensure that the gain of the elements of matrix $[j\omega I-A+BK(j\omega)C]^{-1}\Delta_A$ and vectors $[j\omega I-A+BK(j\omega)C]^{-1}\Delta_B$ and $[j\omega I-A+BK(j\omega)C]^{-1}BK(j\omega)$ tends towards 0 as ω tends towards 0 and infinity to ensure a cancellation of observation error in steady stage and an immunity of this error to measurement noise;
- optimisation of parameters of open loop transmittance (25) through the minimisation of the criterion

$$J=\left\|\left[F_1(j\omega)\quad F_2(j\omega)\quad F_3(j\omega)\right]\right\|_\infty, \quad (48)$$

with

$$F_1(j\omega)=W_A(\omega)[j\omega I-A+BK(j\omega)C]^{-1}\Delta_A$$

$$F_2(j\omega)=W_B(\omega)[j\omega I-A+BK(j\omega)C]^{-1}\Delta_B$$

$$F_3(j\omega)=W_C(\omega)[j\omega I-A+BK(j\omega)C]^{-1}BK(j\omega),$$

where $W_A(\omega)$, $W_B(\omega)$ and $W_C(\omega)$ denotes weighting matrices;

- synthesis of the controller $K(s)$ using the procedure described at the end of section 3.3 (relations (34) and (35)).

5 CONCLUSION

The main contribution of this paper is the development of a dynamic output feedback based observer that will be referred to as a CRONE observer in future developments. This name results in the introduction of CRONE controller in a feedback loop whose goal is to cancel the error between a model state and the unmeasured state of a plant that must be estimated. State observation with a dynamic output feedback based observer is concept that was developed in two papers (Marquez, 2003) and (Marquez and Riaz, 2005). Such an approach of state observation permits:

- a generalisation of the Luenberger form (Luenberger, 1971) that thus allows more freedom and flexibility in the design,
- a formulation allowing a more transparent view of the observer properties in term of feedback elements
- to poses the disturbances rejection problem and the observation robustness problem in the context of robust control theory.

The main differences between this paper and (Marquez, 2003) and (Marquez and Riaz, 2005) are :

- the extension of the dynamic output feedback based observer idea to the observation problem with unknown input,
- the uses of a CRONE controller to solve the disturbances rejection problem and the observation robustness (robustness of the observation error convergence to zero).

With the CRONE controller, plant model perturbations are taken into account in a structured form with no overestimation (but unmodelled dynamics can also be taken into account). Thus, without conservatism introduced in the plant uncertainties modelling, and in spite of a global optimization proof lack of the non convex optimisation problem defined in CRONE control, it turn out that in practice a CRONE controller permits to obtain better performance than an H_∞ one on the same plants (see for instance (Landau, et al, 1995) for a comparison on a benchmark based on robust digital control of a flexible transmission system). Due to the introduction of fractional differentiation, a parameterization of the open loop transfer function with a small number of parameters (three just like a PID controller) is obtained. The optimisation of the control law is thus reduced to the search for the optimal values of these parameters.

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