# PHASE LOCKED LOOPS DESIGN AND ANALYSIS

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Abstract: New methods, for the design of different block diagrams of PLL, using the asymphtotic analysis of high-frequency periodic oscillations, are suggested. The PLL description on three levels is made: 1) on the level of electronic realizations; 2) on the level of phase and frequency relations between inputs and outputs in block diagrams; 3) on the level of differential and integro-differential equations. On the base of such description, the block diagram of floating PLL for the elimination of clock skew and that of frequency synthesizer is proposed. The rigorous mathematical formulation of the Costas loop for the clock oscillators are first obtained. The theorem on a PLL global stability is proved.

# 1 INTRODUCTION

The phase-locked loops are widespread in a modern radio electronics and circuit technology (Viterbi, 1966; Gardner, 1966; Lindsey, 1972; Lindsey and Chie, 1981; Leonov, Reitmann and Smirnova, 1992; Leonov, Ponomarenko and Smirnova, 1996; Leonov and Smirnova, 2000; Kroupa, 2003; Best, 2003, Razavi, 2003; Egan, 2000; Abramovitch, 2002). In this paper the technique of PLL description on three levels is suggested:

1) on the level of electronic realizations,

2) on the level of phase and frequency relations between inputs and outputs in block-diagrams,

3) on the level of differential and integro-differential equations.

The second level, involving the asymptotical analysis of high-frequency oscillations, is necessary for the well-formed derivation of equations and for the passage on the third level of description. For example, the main for the PLL theory notion of phase detector is formed exactly on the second level of consideration. In this case *the characteristic of phase detector depends on the class of considered oscillations*. While in the classical PLL it is used the oscillation multipliers, for harmonic oscillations, the characteristic of phase detector is also harmonic, for the impulse oscillations (for the same electronic realization of feedback loop) it is a continuous piecewise-linear periodic function.

In the present work the development of the above-mentioned technique for PLL is pursued. Here for the standard electronic realizations, the characteristics of phase detectors are computed and the differential equations, describing the PLL operation, are derived.

Here together with usual PLL the Costas loop is also considered. The essential conclusion is that the Costas loop with impulse oscillators tunes to a half frequency of master oscillator.

# 2 BLOCK DIAGRAM AND MATHEMATICAL MODEL OF PLL

Consider a PLL on the first level (Fig.1)

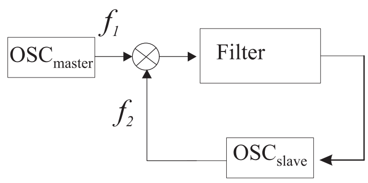


Figure 1: Electronic circuit of PLL.

Here $OSC_{master}$ is a master oscillator, $OSC_{slave}$ is a slave oscillator, which generates high-frequency "almost harmonic oscillations"

$$f_j(t) = A_j \sin(\omega_j(t)t + \psi_j). \qquad (1)$$

Block $\times$ is a multiplier of oscilations of $f_1(t)$ and $f_2(t)$. At its output the signal $f_1(t)f_2(t)$ arises. The relations between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter have the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau.$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial date of filter at the moment $t = 0$.

Now we reformulate the high-frequency property of oscillations $f_j(t)$ to obtain the following condition.

Consider the great fixed time interval $[0,T]$, which can be partitioned into small intervals of the form $[\tau,\tau+\delta], (\tau \in [0,T])$, where the following relations

$$|\gamma(t)-\gamma(\tau)| \le C\delta,\ |\omega_j(t)-\omega_j(\tau| \le C\delta,\quad \forall t \in [\tau,\tau+\delta],\ \forall \tau \in [0,T], \tag{2}$$

$$|\omega_1(\tau)-\omega_2(\tau)| \le C_1,\ \forall \tau \in [0,T], \tag{3}$$

$$\omega_j(t) \ge R,\ \forall t \in [0,T] \tag{4}$$

are satisfied. Here we assume that the quantity $\delta$ is sufficiently small with respect to the fixed numbers $T, C, C_1$, the number $R$ is sufficiently great with respect to the number $\delta$.

The latter means that on the small intervals $[\tau,\tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constants" and the functions $f_j(t)$ rapidly oscillate as harmonic functions. It is clear that such conditions occur for high-frequency oscillations.

Consider two block diagram described in Fig. 2 and 3.

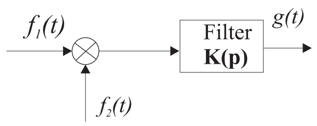


Figure 2: Multiplier and filter with transfer function $K(p)$.

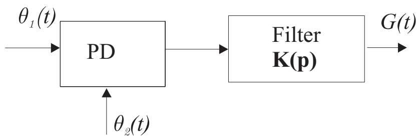


Figure 3: Phase detector and filter.

Here $\theta_j(t) = \omega_j(t)t + \Psi_j$ are phases of the oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$, being called a phase detector (discriminator). The phases $\theta_j(t)$ enter the inputs of PD block and the output is the function $\varphi(\theta_1(t)-\theta_2(t))$.

The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t)-\theta_2(t))$ enter the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$ respectively.

A classical PLL synthesis is based on the following result

*Theorem 1.* If conditions (2)–(4) are satisfied and

$$\varphi(\theta) = \frac{1}{2}A_1A_2\cos\theta,$$

then for the same initial data of filter the following relation

$$|G(t)-g(t)| \le C_2\delta,\ \forall t \in [0,T].$$

is valid. Here $C_2$ is a certain number not depending on $\delta$.

Thus, the outputs of two block-diagrams in Fig. 2 and Fig. 3: $g(t)$ and $G(t)$, differ little from each other and we can pass (from a standpoint of the asymptotic with respect to $\delta$) to the following description level, namely to the level of phase relations 2).

In this case a block diagram in Fig. 1 passes to the following block diagram (Fig. 4)

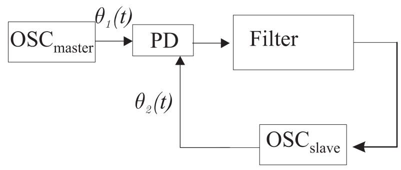


Figure 4: Block diagram of PLL on the level of phase relations.

Consider now the high-frequency oscillators, connected by a diagram in Fig. 1. Here

$$f_j(t) = A_j \mathrm{sign}\ \sin(\omega_j(t)t + \psi_j). \tag{5}$$

We assume, as before, that conditions (2)–(4) are satisfied.

Consider a $2\pi$-periodic function $\varphi(\theta)$ of the form

$$\varphi(\theta) = \begin{cases} A_1A_2(1+2\theta/\pi) \text{ for } \theta \in [-\pi,0], \\ A_1A_2(1-2\theta/\pi) \text{ for } \theta \in [0,\pi]. \end{cases} \tag{6}$$

and block-diagrams in Fig. 2 and 3.

*Theorem 2.* If conditions (2)–(4) are satisfied and the characteristic of phase detector $\varphi(\theta)$ has the form (6), then for the same initial data of filter the following relation holds

$$|G(t)-g(t)| \le C_3\delta,\ \forall t \in [0,T].$$

Here $C_3$ is a certain number not depending on $\delta$.

Theorem 2 is a base for the synthesis of PLL with impulse oscillators. It permits us for the impulse clock oscillators to consider two block-diagrams in parallel: on the level of electronic realization (Fig. 1) and on the level of phase relations (Fig. 4), where the common principles of the phase synchronization theory can be applied. Thus, we can construct the theory of phase synchronization for the distributed system of clocks in multiprocessor cluster.

Let us make a remark necessary to derive the differential equations of PLL.

Consider a quantity

$$\dot{\theta}_j(t) = \omega_j(t) + \dot{\omega}_j(t)t.$$

For the well-synthesized PLL, namely possessing the property of global stability, we have an exponential damping of the quantity $\dot{\omega}_j(t)$:

$$|\dot{\omega}_j(t)| \le Ce^{-\alpha t}.$$

Here $C$ and $\alpha$ are certain positive numbers not depending on $t$. Therefore the quantity $\dot{\omega}_j(t)t$ is, as a rule, sufficiently small with respect to the number $R$ (see condition (2)– (4)).

From the above we can conclude that the following approximate relation

$$\dot{\theta}_j(t) = \omega_j(t) \tag{7}$$

is valid. When derived the differential equations of this PLL, we make use of a block diagram in Fig. 4 and relation (7),which is assumed to be valid precisely .

Note that, by assumption, the control low of tunable oscillators is linear:

$$\omega_2(t) = \omega_2(0) + LG(t). \tag{8}$$

Here $\omega_2(0)$ is the initial frequency of tunable oscillator, $L$ is a certain number, $G(t)$ is a control signal, which is a filter output (Fig. 4).

Thus, the equation of PLL is as follows

$$\dot{\theta}_2(t) = \omega_2(0) + L(\alpha_0(t) + \int_0^t \gamma(t-\tau).\ \varphi(\theta_1(\tau) - \theta_2(\tau))d\tau).$$

Assuming that the master oscillator such that $\omega_1(t) \equiv \omega_1(0)$, we obtain the following relations for PLL

$$(\theta_1(t) - \theta_2(t))^\bullet + L(\alpha_0(t) + \int_0^t \gamma(t-\tau).\ \varphi(\theta_1(\tau) - \theta_2(\tau))d\tau) = \omega_1(0) - \omega_2(0). \tag{9}$$

This is an equation of PLL.

Applying the similar approach, we can conclude that in PLL the filters with transfer functions of more general form can be used:

$$K(p) = a + W(p),$$

where $a$ is a certain number, $W(p)$ is a proper fractional rational function. In this case in place of equation (9) we have

$$(\theta_1(t) - \theta_2(t))^\bullet + L(a(\varphi(\theta_1(t) - \theta_2(t)) + \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta_1(\tau) - \theta_2(\tau))d\tau) = \omega_1(0) - \omega_2(0). \tag{10}$$

In the case when the transfer function of the filter $a + W(p)$ is non degenerate, i.e. its numerator and denominator do not have common roots, equation (10) is equivalent to the following system of differential equations

$$\begin{aligned} \dot{z} &= Az + b\psi(\sigma) \\ \dot{\sigma} &= c^*z + \rho\psi(\sigma). \end{aligned} \tag{11}$$

Here $A$ is a constant $n \times n$-matrix, $b$ and $c$ are constant $n \times n$-vectors, $\rho$ is a number, $\psi(\sigma)$ is a $2\pi$-periodic function, satisfying the relations $\rho = -aL$

$$W(p) = L^{-1}c^*(A - pI)^{-1}b,$$

$$\psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0) - \omega_2(0)}{L(a + W(0))}.$$

Note that in (11) $\sigma = \theta_1 - \theta_2$.

Using Theorem 2, we can make the design of a block diagram of floating PLL, which plays a role of the function of frequency synthesizer and the function of correction of the clock-skew (see parameter $\tau$ in Fig. 5).

Such a block diagram is shown in Fig. 5.

Here $OSC_{master}$ is a master oscillator, *Delay* is a time-delay element, *Filter* is a filter with transfer function

$$W(p) = \frac{\beta}{p + \alpha},$$

$OSC_{slave}$ is a slave oscillator, PD1 and PD2 are programmable dividers of frequencies, *Processor* is a processor.

The *Relay* element plays a role of floating correcting block. The introduction of it allow us to null a residual clock skew, which arises for the nonzero initial difference of frequencies of master and slave oscillators.

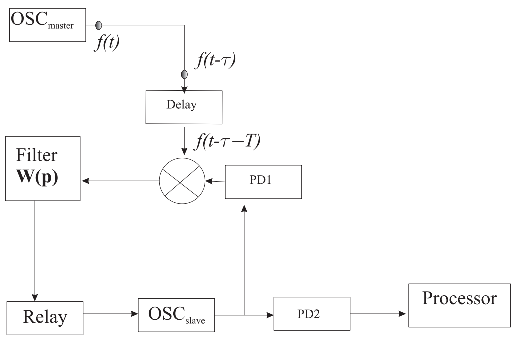


Figure 5: Block diagram of PLL.

Note, the electronic realization of clock and delay can be found in (Ugrumov, 2000; Razavi, 2003) and that of multipliers, filters, and relays in (Aleksenko, 2004; Razavi, 2003). The description of dividers of frequency can be found in (Solonina et al., 2000).

Assume, as usual, that the frequency of master oscillator is constant, namely $\omega_1(t) \equiv \omega_1 = const$. The parameter of delay line $T$ is chosen in such a way that $\omega_1(T+\tau) = 2\pi k + 3\pi/2$. Here $k$ is a certain natural number, $\omega_1\tau$ is a clock skew.

By Theorem 2 and the choice of $T$ the block diagram, shown in Fig. 6, can be changed by the close block diagram, shown in Fig. 6.

Here $2\pi$ is a periodic characteristic of phase detector. It has the form

$$\varphi(\theta) = \begin{cases} 2A_1A_2\theta/\pi & \text{for}\,\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 2A_1A_2(1-\theta/\pi) & \text{for}\,\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}], \end{cases} \tag{12}$$

$\theta_2(t) = \theta_3(t)/M$, $\theta_4(t) = \theta_3(t)/N$, where the natural numbers $M$ and $N$ are the parameters of programmable divisions PD1 and PD2.

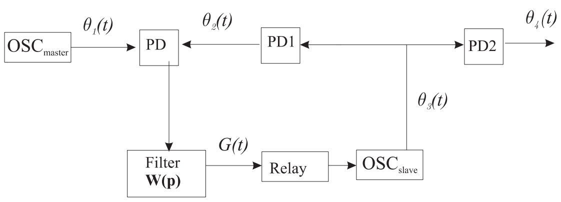


Figure 6: Equivalent block diagram of PLL.

For transient process (capture mode) the following conditions

$$\lim_{t\to+\infty}(\theta_4(t) - \frac{M}{N}\theta_1(t)) = \frac{2\pi k M}{N} \tag{13}$$

(phase capture)

$$\lim_{t\to+\infty}(\dot\theta_4(t) - \frac{M}{N}\dot\theta_1(t)) = 0 \tag{14}$$

(frequency capture) must be satisfied.

Relations (13) and (14) are the main requirements of PLL for array processors. The time of transient processors depends on the initial data and is sufficiently large for multiprocessors system (Leonov and Seledzhi, 2002; Kung, 1988). Here a difference between the beginning of transient process and the beginning of performance of parallel algorithm can be some minutes. This difference is very large for the electronic systems.

Assuming that the characteristic of relay is of the form $\Psi(G) = \text{sign}G$ and the actuating element of slave oscillator is linear, we have

$$\dot\theta_3(t) = R\,\text{sign}G(t) + \omega_3(0), \tag{15}$$

where $R$ is a certain number, $\omega_3(0)$ is the initial frequency, $\theta_3(t)$ is a phase of slave oscillator.

Taking into account relations (15), (1), (12), and the block diagram in Fig. 6, we have the following differential equations of PLL

$$\begin{aligned} &\dot G + \alpha G = \beta\varphi(\theta) \\ &\dot\theta = -\frac{R}{M}\text{sign}G + (\omega_1 - \frac{\omega_3(0)}{M}). \end{aligned} \tag{16}$$

Here $\theta(t) = \theta_1(t) - \theta_2(t)$.

## 3 CRITERION OF GLOBAL STABILITY OF PLL

System (16) can be written as

$$\begin{aligned} &\dot G = -\alpha G + \beta\varphi(\theta) \\ &\dot\theta = -F(G), \end{aligned} \tag{17}$$

where

$$F(G) = \frac{R}{M}\text{sign}G - (\omega_1 - \frac{\omega_3(0)}{M}).$$

*Theorem 3.* If the inequality

$$|R| > |M\omega_1 - \omega_3(0)| \tag{18}$$

is valid, then any solution of system (17) as $t \to +\infty$ tends to a certain equilibrium.

If the inequality

$$|R| < |M\omega_1 - \omega_3(0)| \tag{19}$$

is valid, then all the solutions of system (17) tends to infinity as $t \to +\infty$.

Consider the equilibria for system (17). For any equilibrium we have

$$\dot\theta(t) \equiv 0, \quad G(t) \equiv 0, \quad \theta(t) \equiv \pi k.$$

Theorem 4. Let relation (18) be valid. In this case, if $R > 0$, then the following equilibria

$$G(t) \equiv 0, \quad \theta(t) \equiv 2k\pi \tag{20}$$

are locally asymptotically stable and the following equilibria

$$G(t) \equiv 0, \quad \theta(t) \equiv (2k+1)\pi \tag{21}$$

are locally unstable. If $R < 0$, then equilibria (21) are locally asymptotically stable and equilibria (20) are locally unstable.

Thus, for relations (13) and (14) to be satisfied it is necessary to choice the parameters of system in such a way that the inequality holds

$$R > |M\omega_1 - \omega_3(0)|. \tag{22}$$

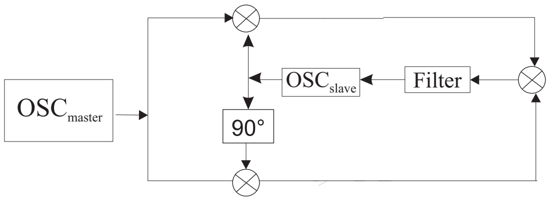


Figure 7: Costas loop.

# 4 COSTAS LOOP

Consider now a block diagram of the Costas loop (Fig. 7)

Here all denotations are the same as in Fig. 1, $90^o$ – is a quadrature component. As before, we consider here the case of the high-frequency harmonic and impulse signals $f_j(t)$.

However together with the assumption that conditions (2) and (4) are valid we assume also that (3) is satisfied for the signal of the type (1) and the relation

$$|\omega_1(\tau) - 2\omega_2(\tau)| \leq C_1, \quad \forall \tau \in [0, T], \tag{23}$$

is valid for the signal of the type (5).

Applying the similar approach we can obtain differential equation for the Costas loop, where

$$\begin{aligned} \dot{z} &= Az + b\Psi(\sigma) \\ \dot{\sigma} &= c^* z + \rho\Psi(\sigma). \end{aligned} \tag{24}$$

Here $A$ is a constant $n \times n$-matrix, $b$ and $c$ are constant $n$-vectors, $\rho$ is a number, $\Psi(\sigma)$ is a $2\pi$-periodic function, satisfying the following relations

$$\rho = -2aL, \quad W(p) = (2L)^{-1} c^* (A - pI)^{-1} b,$$

$$\psi(\sigma) = \frac{1}{8} A_1^2 A_2^2 \sin\sigma - \frac{\omega_1(0) - \omega_2(0)}{L(a + W(0))},$$

$$\sigma = 2\theta_1 - 2\theta_2 \quad \text{(in the case (1);}$$

$$\psi(\sigma) = P(\sigma) - \frac{\omega_1(0) - 2\omega_2(0)}{2L(a + W(0))},$$

$$P(\sigma) = \begin{cases} -2A_1^2 A_2^2 \left(1 + \frac{2\sigma}{\pi}\right), & \sigma \in [0, \pi] \\ -2A_1^2 A_2^2 \left(1 - \frac{2\sigma}{\pi}\right), & \sigma \in [-\pi, 0] \end{cases}$$

$$\sigma = \theta_1 - 2\theta_2 \quad \text{(in the case (5).}$$

From the above equations it follows that for deterministic (when the noise is lacking) description of the Costas loops the conventional introduction of additional filters turns out unnecessary. Here a central filter plays their role.

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