ASYMPTOTIC THEORY OF THE REACHABLE SETS TO LINEAR PERIODIC IMPULSIVE CONTROL SYSTEMS

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Abstract: We study linear periodic control systems with a bounded total impulse of control. The main result is an asymptotic formula for the reachable set, which, at the same time, reveals the structure of the attractor — the set of all limit shapes of the reachable sets. The attractor is shown to be parameterized by a (finite-dimensional) toric fibre bundle over a circle. The fibre of the bundle can be described via the Floquet multipliers (monodromy matrix) of the linear system. Moreover, the limit dynamics of shapes of reachable sets can be parametrized by an explicit curve on the toric bundle.

1 INTRODUCTION

One of the fundamental notions of control theory is that of reachable sets which provide a visible bound for control capabilities. In general, these sets have a complicated shape and dynamics. There is, however, a kind of problems where the behavior of reachable sets is well-understood.

Namely, it turns out that the reachable sets of linear control systems have simple limit properties as time evolves to infinity provided that a suitable time-dependent matrix scaling is applied. This kind of results was found for the first time in (Ovseevich, 1991), where time-invariant linear control systems with geometric bounds on control were studied. It was shown that in this setup there is a single limit shape of the reachable sets, shape being the set regarded up to an arbitrary nondegenerate linear transform.

At present the scope of this phenomena is not yet clear cut. It is very likely that there is a natural extension of these results to general time-dependent linear systems. Moreover, a similar phenomena was discovered for some nonlinear stochastic dynamic systems (Dolgopyat et al., 2004).

The purpose of the study is to develop the asymptotic theory of the reachable sets to linear impulsive control systems. A motivation to address impulsive control systems is also due to the perceived relevance of the impulsive control theory for hybrid systems whose state evolution is dictated by the interaction of conventional time-driven dynamics and event-driven dynamics (see, e.g., (Aubin, 2000; Branicky et al., 1998; Miller and Rubinovich, 2003)).

In this paper, we study the periodic linear control systems with a bounded total impulse of control. The main result is an asymptotic formula for the reachable sets (see (3)), that, in particular, reveals the structure of attractor — the set of all limit shapes of the reachable sets.

It would be extremely interesting to understand the limit behavior of reachable sets for a general linear system. Unfortunately, the nature of the present methods is computational and it looks like new ideas are needed in order to grasp the limit dynamics of the reachable sets.

2 PROBLEM STATEMENT

Consider a linear control system on the time interval $[0,T]$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(0) = 0, \quad (1)$$

under the constraint on the total impulse of control u:

$$\int_0^T \langle f(t), u(t)dt\rangle \le 1, \qquad (2)$$

where $x(t) \in \mathbb{V} = \mathbb{R}^n$, $u(t) \in \mathbb{W} = \mathbb{R}^m$, $A(t)$, $B(t)$ are matrices of appropriate dimensions, $U(t)$ is a given central symmetric convex body in $\mathbb{W}$, f is an arbitrary continuous function such that $f(t) \in U^\circ(t)$, and $U^\circ(t)$ is the polar of set $U(t)$.

Assume that the Kalman type condition of complete controllability holds, namely, for any vector $u \in \mathbb{W}$ and a time moment T, function $\Phi(T,t)B(t)u$ does not vanish identically in any interval of time t. Under these assumptions the reachable sets $\mathcal{D}(T)$ to system (1), (2) are central symmetric convex bodies.

The problem addressed is to study the limit behavior of the reachable sets $\mathcal{D}(T)$ as $T \to \infty$. The reachable sets are regarded as elements of the metric space $\mathbb{B}$ of central symmetric convex bodies with the Banach-Mazur distance ρ:

$$\rho(\Omega_1, \Omega_2) = \log(t(\Omega_1, \Omega_2)t(\Omega_2, \Omega_1)),$$
$$\text{where } t(\Omega_1, \Omega_2) = \inf\{t \ge 1 : t\Omega_1 \supset \Omega_2\}.$$

The general linear group $GL(\mathbb{V})$ naturally acts on the space $\mathbb{B}$ by isometries. The factorspace $\mathbb{S}$ is called the space of shapes of central symmetric convex bodies, where the shape $\mathrm{Sh}\,\Omega \in \mathbb{S}$ of a convex body $\Omega \in \mathbb{B}$ is the orbit $\mathrm{Sh}\,\Omega = \{C\Omega : \det C \ne 0\}$ of the point Ω with respect to the action of $GL(\mathbb{V})$. The Banach-Mazur factormetric makes $\mathbb{S}$ into a compact metric space. The convergence of the reachable sets $\mathcal{D}(T)$ and their shapes is understood in the sense of the Banach-Mazur metric. For two asymptotically equal functions with values in the space of convex bodies or the space of their shapes, the following notations are used: $\Omega_1(T) \sim \Omega_2(T)$, if $\rho(\Omega_1(T), \Omega_2(T)) \to 0$ as $T \to \infty$, and similarly $\mathrm{Sh}\,\Omega_1(T) \sim \mathrm{Sh}\,\Omega_2(T)$, if $\rho(\mathrm{Sh}\,\Omega_1(T), \mathrm{Sh}\,\Omega_2(T)) \to 0$ as $T \to \infty$. The convergence of convex bodies may be also understood in the sense of convergence of their support functions. Remind that the support function of a convex compact set is given by formula: $H_\Omega(\xi) = \sup_{x\in\Omega}\langle x, \xi\rangle$, where $\xi \in \mathbb{V}^*$, and uniquely defines the set Ω. The equivalence of the two definitions of convergence of convex bodies — in the terms of convergence of their support functions and in the sense of the Banach-Mazur metric — is established by the following lemma (Figurina and Ovseevich, 1999):

Lemma 1. *A sequence $\Omega_i \in \mathbb{B}$ converges to $\Omega \in \mathbb{B}$ in the sense of the Banach-Mazur metric if and only if the corresponding sequence of the support functions $H_i(\xi) = H_{\Omega_i}(\xi)$ converges to the support function $H_\Omega(\xi)$ pointwise and is uniformly bounded on the unit sphere in the dual space $\mathbb{V}^*$.*

We address the periodic case, when the constituents A, B, and U of control system (1), (2) are supposed to be continuous and periodic in t. To fix ideas, the period is assumed to be 1. It would be interesting to understand the limit behavior of reachable sets for a general linear system. This problem, however, seems rather difficult, since already the time-invariant case is nontrivial, and, say, for quasi-periodic systems it is not clear how to prove the corresponding natural conjectures.

3 ASYMPTOTIC BEHAVIOR OF SHAPES OF THE REACHABLE SETS

We study the limit behavior as $T \to +\infty$ of the curve $T \mapsto \mathrm{Sh}\,\mathcal{D}(T)$ under different assumptions on the spectrum of the monodromy matrix. At the heart of the considerations below there is an explicit formula for the support function of the reachable set:

Lemma 2. *The support function of the reachable set $\mathcal{D}(T)$ to system (1), (2) is given by*

$$H_{\mathcal{D}(T)}(\xi) = \sup_{t\in[0,T]} H_{U(t)}\left(B(t)^*\Phi(T,t)^*\xi\right), \qquad (3)$$

where $\Phi(t,s)$ is the fundamental matrix of linear system $\dot{x} = A(t)x$.

Stable Case. Let the system (1) be asymptotically stable, i.e.

$$\Phi(T,t) = o(1) \text{ as } T - t \to +\infty,$$

and $o(1)$ is uniformly small. It is easy to establish the stability criterion: system (1) is asymptotically stable iff the spectrum of the monodromy matrix $M = \Phi(1,0)$ is contained in the open unit disk of the complex plane.

Let us show that the curve $T \mapsto \mathcal{D}(T)$ is asymptotically periodic as $T \to \infty$. In other words, there exists such a continuous periodic curve $f : \mathbb{R}/\mathbb{Z} \to \mathbb{B}$ that $\mathcal{D}(T) \sim f(T)$ as $T \to +\infty$. Informally speaking, the curve $T \mapsto \mathcal{D}(T)$ is reeled on a limit cycle.

The curve f can be given by an explicit formula. Define function

$$\mathcal{F}(T) = \mathcal{F}(T,\xi) = \sup_{t\in(-\infty,T]} H_U\left(B^*\Phi(T,t)^*\xi\right), \qquad (4)$$

where the argument t of periodic functions B and U is omitted. Due to the stability condition, $\mathcal{F}(T)$ is a continuous periodic function of T. The periodicity of $\mathcal{F}$ follows from the equality

$$\Phi(T+1, t+1) = \Phi(T,t)$$

for the fundamental matrix of a 1-periodic system. Furthermore, for each T, function $\xi \mapsto \mathcal{F}(T,\xi)$ is homogeneous and convex, and therefore it is the support function of a body $f(T)$. Thus, the curve f is defined. On the other hand, from (3), (4) it follows that

$$H_{f(T)}(\xi) = H_{\mathcal{D}(T)}(\xi) \text{ for large enough } T, \quad (5)$$

since $\Phi(T,t) = O\left(e^{-\beta(T-t)}\right)$, where $\beta > 0$ owing to the assumed stability property. The asymptotic equality $\mathcal{D}(T) \sim f(T)$ follows from (5).

Unstable Case. Assume that the system (1) is strictly unstable, i.e.

$$\Phi(T,s) = o(1) \text{ as } T - s \to -\infty,$$

where $o(1)$ is uniformly small. System (1) is strictly unstable iff the spectrum of the monodromy matrix $M = \Phi(1,0)$ is contained in the complement of the closed unit disk of the complex plane.

Define the matrix multiplier $C(T) = \Phi(0,T)$ and consider the set

$$\widetilde{\mathcal{D}}(T) \stackrel{\text{def}}{=} C(T)\mathcal{D}(T).$$

It is easy to see that

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = \sup_{t\in[0,T]} H_U\left(B^*\Phi(0,t)^*\xi\right),$$

and from the instability criterion it follows that $\Phi(0,t)$ decreases exponentially fast as $t \to \infty$. Therefore, the values $\sup_{t\in[0,T]} H_U\left(B^*\Phi(0,t)^*\xi\right)$ converge as $T \to \infty$, and the convergence of bodies $\widetilde{\mathcal{D}}(T) \to \mathcal{D}_\infty$, takes place, where

$$H_{\mathcal{D}_\infty}(\xi) = \sup_{t\in[0,\infty]} H_U\left(B^*\Phi(0,t)^*\xi\right).$$

Thus, we have the asymptotical equality:

$$\mathcal{D}(T) \sim \Phi(T,0)\mathcal{D}_\infty.$$

In the view of the behavior of shapes of the reachable sets, there is an essential difference between stable and unstable cases. In the unstable case, shapes $\mathrm{Sh}\,\mathcal{D}(T)$ converge as $T \to \infty$, while, for stable system, the curve $T \to \mathrm{Sh}\,\mathcal{D}(T)$ is reeled on a limit cycle.

Note that the above considerations admit an extension to almost periodic systems.

Yet another choice $C(T) = \Phi(\{T\},T)$, where $\{T\}$ is the fractional part of T, of a normalizing matrix factor seems also reasonable. It is easy to see at that rate, that the limit normalized body $\mathcal{D}_\infty = \mathcal{D}_\infty(T)$ depends on time periodically. This choice of the matrix factor fits better the general case, when stable, unstable and neutral components are present.

Neutral Case. Assume that system (1) is neutral, meaning that the spectrum of the monodromy matrix $M = \Phi(1,0)$ rests on the unit circle. Consider the Jordan decomposition

$$M = U\mathfrak{D} = e^{N+D},$$

where $\mathfrak{D}$ is a diagonalizable matrix of the same spectrum that the matrix M has, D is such a diagonalizable real matrix that $\mathfrak{D} = e^D$, N is a nilpotent matrix, and $ND = DN$. As is well known, there exists a matrix $F = F(N,T)$ with the following properties:

$$FNF^{-1} = T^{-1}N,\ FD = DF,$$
$$\text{and } F_\infty = F_\infty(N) = \lim_{T\to+\infty} F(N,T) \text{ is defined.}$$

Put

$$N(T) = \Phi(T,0)N\Phi(0,T),\ D(T) = \Phi(T,0)D\Phi(0,T).$$

It is easy to see that $N(T)$ and $D(T)$ are periodic functions of T, since matrices N and D commute with $M = \Phi(1,0)$. Matrix function

$$F_\infty(N(T)) = \Phi(T,0)F_\infty(N)\Phi(0,T)$$

is also continuous and periodic. Function ϕ given by formula

$$\phi(T,t) = e^{(t-T)[N(T)+D(T)]}\Phi(T,t), \quad (6)$$

is periodic in T and t. Define the matrix factor

$$C(T) = F(N(T),T)e^{T[N(T)+D(T)]}$$

and consider the normalized set

$$\widetilde{\mathcal{D}}(T) \stackrel{\text{def}}{=} C(T)\mathcal{D}(T).$$

It is easy to see that

$$\begin{aligned} H_{\widetilde{\mathcal{D}}(T)}(\xi) &= \sup H\left(g_T^1(t)\right) = \sup H\left(g_T^2(t)\right) \\ &= \sup H\left(g_T(t)\right) + o(1), \end{aligned} \quad (7)$$

where sup is taken over $t \in [0,T]$, $H = H_U$ and

$$g_T^1(t) = B^*\phi(T,t)^*e^{t(N^*(T)+D^*(T))}F(N(T),T)^*\xi,$$
$$g_T^2(t) = B^*\phi(T,t)^*F(N(T),T)^*e^{\frac{t}{T}N^*(T)+tD^*(T)}\xi,$$
$$g_T(t) = B^*\phi(T,t)^*F_\infty(N(T))^*e^{\frac{t}{T}N^*(T)+tD^*(T)}\xi.$$

The last term $o(1)$ in (7) appears on account of the difference between $F(T)$ and $F_\infty(N(T))$. Since other matrices involved in $g_T^2(t)$ are uniformly bounded, the difference of the arguments comes to $o(1)$.

Consider the following function of two arguments T and L:

$$I(L,T) = \sup_{t\in[0,L]} f_T(t,\mathbf{t},\tau) =$$
$$= \sup_{t\in[0,L]} H_U\left(B^*\phi(T,t)^*F_\infty(N(T))^*e^{\frac{t}{L}N^*(T)+tD^*(T)}\xi\right).$$

Function $f_T(t,\mathbf{t},\tau)$, where $\mathbf{t}=e^{tD(T)}$, $\tau=t/L$, is periodic in t and depends on the parameter T periodically as well. By using the Hermann Weyl averaging method (Weyl, 1938; Weyl, 1939; Arnold, 1989), like in (Goncharova and Ovseevich, 2007), we obtain the asymptotic representation

$$I(L,T)=\sup_{\mathcal{T}\times J} f_T(t,\mathbf{t},\tau)+o(1) \qquad (8)$$

as $L\to\infty$, where $o(1)$ is small uniformly in T. In formula (8), the interval $J=[0,1]$ and a torus $\mathcal{T}=\mathcal{T}(T)$ are involved. The torus is the closure of the one-parameter subgroup $\{(e^{2\pi it},e^{tD(T)})\}$ in the group $S^1\times \mathrm{GL}(\mathbb{V})$. Notice that the torus

$$\mathcal{T}(T)=\Phi(T,0)\mathcal{T}(0)\Phi(0,T)$$

depends on T continuously and periodically. The torus can be naturally represented as a fibre bundle over the circle, at that the fibre over $e^{2\pi iT}\in S^1$ is the closure of the cyclic group generated by matrix $e^{D(T)}\in \mathrm{GL}(\mathbb{V})$.

It is clear that

$$I(T)=\sup_{\mathcal{T}\times J} f_T(t,\mathbf{t},\tau)$$

is a periodic function of parameter T. Hence, for large L,

$$I(L,T)=I(T)+o(1)$$

is a periodic function of T up to $o(1)$. In particular, this is true for $L=T$ and large T. From this we conclude that the curve $T\mapsto \mathrm{Sh}\,\mathcal{D}(T)$ is periodic up to $o(1)$, i.e. it is reeled on a limit cycle.

Stable-neutral Case. Suppose that system (1) is stable-neutral. This means that fundamental matrix admits a polynomial estimate:

$$|\Phi(T,t)|=O(1+|T-t|^n) \ \text{ as } T-t\to+\infty.$$

It is not difficult to obtain the criterion of stable-neutrality: system (1) is stable-neutral iff the spectrum of the monodromy matrix $M=\Phi(1,0)$ is contained in the closed unit disk of the complex plane.

Consider the canonical decomposition of the monodromy matrix $M=\Phi(1,0)$

$$M=M_0\oplus M_-$$

into the stable and neutral components (in accordance with the relations $|\lambda|<1$, $|\lambda|=1$ for eigenvalues), and the corresponding decomposition of phase space:

$$\mathbb{V}=\mathbb{V}_0\oplus\mathbb{V}_-.$$

For an arbitrary time moment T, the monodromy matrix

$$M(T)=\Phi(T+1,T)=\Phi(T,0)M\Phi(0,T)$$

depends on T periodically. The corresponding decomposition

$$\mathbb{V}(T)=\mathbb{V}_0(T)\oplus\mathbb{V}_-(T).$$

is also periodic in T.

The scaling matric factor $C(T)$ can be taken in the block-diagonal form

$$C(T)=C_0(T)\oplus C_-(T),$$

where $C_i(T):\mathbb{V}_i(T)\to\mathbb{V}_i(T)$ are given by formulas

$$C_0(T)=F(N(T),T),\ C_-(T)=I.$$

The support function $H_{\widetilde{\mathcal{D}}(T)}(\xi)$ of the normalized body $\widetilde{\mathcal{D}}(T)\overset{\text{def}}{=}C(T)\mathcal{D}(T)$ is as follows

$$H_{\widetilde{\mathcal{D}}(T)}(\xi)=\sup_{t\in[0,T]} f_T(t)=\sup_{t\in[0,T]} H_U(g_T(t)), \qquad (9)$$

$$g_T(t)=B^*\Phi(T,t)^*\xi_-+$$

$$B^*\phi(T,t)^*e^{(T-t)(N^*(T)+D^*(T))}F(N(T),T)^*\xi_0,$$

$\xi_i=\xi_i(T)\in\mathbb{V}_i(T)^*$, $i\in\{-,0\}$ are components of the canonical decomposition of a vector $\xi\in\mathbb{V}^*$, and the periodic in both arguments function $\phi(T,t)$ is defined in (6). Notice that in the generic case, matrix F is the identity one, and therefore, the formula for the support function is just as simple as (3):

$$H_{\widetilde{\mathcal{D}}(T)}(\xi)=\sup_{t\in[0,T]} H_U\left(B^*\Phi(T,t)^*\xi\right).$$

To study $H_{\widetilde{\mathcal{D}}(T)}(\xi)$ as $T\to\infty$ let us apply the decomposition method like in the autonomous case (Goncharova and Ovseevich, 2007). Owing to the basic commutativity relations for matrices F, N, and D, we have

$$e^{(T-t)(N^*(T)+D^*(T))}F(N(T),T)^*\xi_0=$$
$$=F(N(T),T)^*e^{\frac{T-t}{T}N^*(T)+(T-t)D^*(T)}\xi_0,$$

and, thus, we have the uniform asymptotic equality

$$\Phi(T,t)^*F(N(T),T)^*\xi_0=$$
$$\phi(T,t)^*F_\infty(N(T))^*e^{\frac{T-t}{T}N^*(T)+(T-t)D^*(T)}\xi_0+o(1)$$

as $T\to\infty$. Like in (Goncharova and Ovseevich, 2007), we divide the time interval $\mathcal{I}=[0,T]$ into the two subintervals

$$\mathcal{I}=\mathcal{I}_0\cup\mathcal{I}_-=[0,(1-\varepsilon)T]\cup[(1-\varepsilon)T,T],$$

where $\varepsilon=\varepsilon(T)$ is such that $\varepsilon(T)=o(1)$, while $\varepsilon(T)T\to\infty$ as $T\to\infty$. By adopting arguments from (Goncharova and Ovseevich, 2007), we obtain the asymptotic equality

$$H_{\widetilde{\mathcal{D}}(T)}(\xi)=\max\{H_{-0}(T,\xi),H_0(T,\xi)\}+o(1), \qquad (10)$$

where in accordance with notations (9)

$$H_{-0} = \sup_{t\in I_-} f_T(t) + o(1) = \sup_{t\in(-\infty,T]} H_U(g_{-0T}(t)),$$

where $g_{-0T}(t)$ stands for

$$B^*\Phi(T,t)^*\xi_- + B^*\phi(T,t)^*F_\infty(N(T))^*e^{(T-t)D^*(T)}\xi_0,$$

while

$$H_0 = \sup_{t\in I_0} f_T(t) + o(1) = \sup_{t\in\mathbf{R},\tau\in[0,1]} H_U(g_{0T}),$$

where $g_{0T}(t,\tau)$ stands for

$$B^*\phi(T,t)^*F_\infty(N(T))^*e^{\tau N^*(T)+(T-t)D^*(T)}\xi_0.$$

Functions $H_{-0}(T,\xi)$ and $H_0(T,\xi)$ are periodic in T, and convex, homogeneous in ξ. From this it follows that $H_i(T,\xi)$, $i=-0,0$, are the support functions of some convex compact sets $\Omega_{-0}(T)\subset\mathbb{V}(T)$ and $\Omega_0(T)\subset\mathbb{V}_0(T)$, which periodically depend on T. Furthermore, the convex compact

$$\Omega(T) = \Omega_{-0}(T)*\Omega_0(T)\subset\mathbb{V}(T),$$

and $\Omega_0(T))$ is a body with the support function

$$\max\{H_{-0}(T,\xi),H_0(T,\xi)\},$$

and also periodically depends on T. Here, we use the join notation:

$$\Omega = \Omega'*\Omega'',$$

meaning that Ω is the convex hull of the union $\Omega'\bigcup\Omega''$, or what is the same $H_\Omega = \max\{H_{\Omega'},H_{\Omega''}\}$. Thus, the curve $T\mapsto\widetilde{\mathcal{D}}(T)$ is reeled on the limit cycle $T\mapsto\Omega(T)$.

Unstable-neutral Case. Similarly to stable-neutral case, fundamental matrix admits a polynomial estimate

$$|\Phi(T,t)| = O(1+|T-t|^n) \text{ as } T-t\to-\infty.$$

System (1) is unstable-neutral iff the spectrum of the monodromy matrix $M=\Phi(1,0)$ is contained in the closed complement of the unit disk of the complex plane. In this case, the normalizing matrix factor $C(T)$ can be taken in the block-diagonal form:

$$C(T) = C_0(T)\oplus C_+(T),$$

where $C_i(T):\mathbb{V}_i(T)\to\mathbb{V}_i(T)$ are defined by formulas

$$C_0(T) = F(N(T),T),\; C_+(T) = \Phi(\{T\},T).$$

We note in passing that if we put $C_+(T)=\Phi(0,T)$, then the block-diagonal structure of the normalizing matrix would be lost, since, in general, $\Phi(0,T)$ do not map $\mathbb{V}_+(T)$ into itself.

The support function $H_{\widetilde{\mathcal{D}}(T)}(\xi)$ of the normalized body $\widetilde{\mathcal{D}}(T)\overset{\text{def}}{=}C(T)\mathcal{D}(T)$ is similar to (9):

$$\sup_{t\in[0,T]} H_U(B^*\Phi(\{T\},t)^*\xi_+ + B^*\phi(T,t)^*\times$$
$$\times e^{(T-t)(N^*(T)+D^*(T))}F(N(T),T)^*\xi_0),$$

and like in (10) we have the asymptotics

$$H_{\widetilde{\mathcal{D}}(T)}(\xi) = \max\{H_{+0}(T,\xi),H_0(T,\xi)\}+o(1),$$

where $H_{+0}(T,\xi) = \sup_{t\in[0,\infty)} H_U(g_{+0T}(t))$, and

$$g_{+0T}(t) = B^*\Phi(\{T\},t)^*\xi_+ +$$
$$+B^*\phi(T,t)^*F_\infty(N(T))^*e^{N(T)^*}e^{(T-t)D^*(T)}\xi_0.$$

The essential difference, in contrast to stable-neutral situation, consists of that, on this occasion, function $H_{+0}(T,\xi)$ is not periodic in T, however, it becomes periodic by the transformation of variable $\xi_0\mapsto e^{TD^*(T)}\xi_0$. Geometrically, this means that the asymptotic equality

$$\widetilde{\mathcal{D}}(T)\sim p(\mathbf{t}_T)\Omega_{+0}(T)*\Omega_0(T) \text{ as } T\to\infty,$$

holds, where $\mathbf{t}_T = (e^{2\pi iT},e^{TD(T)})$ is an element of the torus $\mathcal{T}(T)$, matrix $p(\mathbf{t}_T)=e^{TD(T)}$ is the second component of $\mathbf{t}_T$, $\Omega_\alpha(T)$ are periodically depending on time convex compacts in spaces $V_\alpha(T)$, $\alpha\in\{+0,0\}$.

Thus, the limit behavior of the normalized reachable sets $\widetilde{\mathcal{D}}(T)$ is the same as the behavior of the curve $T\mapsto\mathbf{t}_T$. The closure of the curve might have an arbitrary large dimension so that the curve is reeled on a multidimensional manifold. Still, the shapes of the reachable sets $\mathrm{Sh}\,\mathcal{D}(T)$ have a simpler behavior, since $\mathrm{Sh}\,\mathcal{D}(T)\sim\mathrm{Sh}\,(\Omega_{+0}(T)*\Omega_0(T))$, and, therefore, the curve $T\mapsto\mathrm{Sh}\,\mathcal{D}(T)$ is reeled on a limit cycle of dimension not greater than 1.

General Case. The general result can be obtained by using the decomposition method (see (Goncharova and Ovseevich, 2007)) and the considered above cases. Consider the canonical decomposition of the monodromy matrix $M=\Phi(1,0)$

$$M = M_+\oplus M_0\oplus M_-$$

into the unstable, neutral, and stable components (in accordance with the relations $|\lambda|<1$, $|\lambda|>1$, $|\lambda|=1$ for eigenvalues), and the corresponding decomposition of phase space

$$\mathbb{V} = \mathbb{V}_+\oplus\mathbb{V}_0\oplus\mathbb{V}_-.$$

For an arbitrary time moment T, the monodromy matrix

$$M(T) = \Phi(T+1,T) = \Phi(T,0)M\Phi(0,T)$$

depends on T periodically, so does the corresponding decomposition of phase space

$$\mathbb{V} = \mathbb{V}_+(T) \oplus \mathbb{V}_0(T) \oplus \mathbb{V}_-(T).$$

In the general case, the scaling matric factor $C(T)$ can be chosen in the block-diagonal form

$$C(T) = C_+(T) \oplus C_0(T) \oplus C_-(T),$$

where $C_i(T) : \mathbb{V}_i(T) \to \mathbb{V}_i(T)$ are given by formulas

$$C_+(T) = \Phi(\{T\}, T),$$
$$C_0(T) = F(N(T), T),\ C_-(T) = I.$$

The normalized body $\widetilde{\mathcal{D}}(T) = C(T)\mathcal{D}(T)$ has the following asymptotics

$$\widetilde{\mathcal{D}}(T) \sim p(\mathbf{t})\Omega_{+0}(T) * \Omega_0(T) * \Omega_{-0}(T) \qquad (11)$$

as $T \to \infty$, where $\mathbf{t} = (e^{2\pi iT}, e^{TD(T)})$ is an element of the torus $\mathcal{T}(T)$, matrix $p(\mathbf{t}) = e^{TD(T)}$ is the second component of $\mathbf{t}$, $\Omega_\alpha(T)$ are periodically depending on time convex compacts in $V_\alpha(T)$, $\alpha \in \{+0, 0\}$. Asymptotic equality (11) can be naturally interpreted in the terms of attractors in space $\mathbb{S}$ of shapes of convex bodies. Define a fibre bundle over circle $\mathcal{P} \to S^1$ as follows:

$$\mathcal{P} = \{(e^{2\pi iT}, e^{TD(T)}z) \in S^1 \times \mathrm{GL}(\mathbb{V}) : z \in \mathcal{Z}(T)\},$$
$$(e^{2\pi iT}, e^{TD(T)}z) \mapsto e^{2\pi iT},$$

where $\mathcal{Z}(T)$ is the closure in $\mathrm{GL}(\mathbb{V})$ of a cyclic group generated by matrix $e^{D(T)}$. Then, the relation (11) asserts that the totality $\mathcal{A}$ of all the limit shapes of the reachable sets (attractor) is parameterized by the set $\mathcal{P}$: there is a continuous map σ from $\mathcal{P}$ onto $\mathcal{A}$. At that, in the limit, the curve $T \mapsto \mathrm{Sh}\,\mathcal{D}(T)$ is parameterized by the curve $T \mapsto \mathbf{t}_T = (e^{2\pi iT}, e^{TD(T)})$ in $\mathcal{P}$ in the sense that

$$\mathrm{Sh}\,\mathcal{D}(T) \sim \sigma(\mathbf{t}_T)$$

as $T \to \infty$.

This asymptotic equality is an incarnation of (11) and the main result of the paper. It says that the limit dynamics of the shapes $\mathrm{Sh}\,\mathcal{D}(T)$ can be described via the “straight winding” $T \mapsto \mathbf{t}_T$ in the toric bundle $\mathcal{P}$.

4 CONCLUSIONS

In this paper we determined completely the asymptotic behavior of reachable sets to periodic linear dynamic systems with impulsive control. This is just a single step in the long road directed to understanding the limit behavior of reachable sets for general linear systems. Still, our results for the periodic case suggest a reasonable conjectural description of this behavior. In fact, it is possible to state a precise conjecture pertaining to the quasi-periodic case.

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