VARIATIONAL DATA ASSIMILATION FOR MISSING DATA INTERPOLATION IN SST IMAGES

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1. THE PROBLEM

Because of the cloud coverage, sea surface temperature (SST) images produced from satellite recordings contain missing data. Usually, a preliminary step before any processing on the SST images is the missing data interpolation. Many algorithms have already been proposed for missing data interpolation. Up to our knowledge, most of the proposed methods are based on the Kalman Filtering or smoothing framework [1]. The Kalman filters are probabilistic methods to estimate the optimal state sequence of a linear system defined by uni-modal Gaussian distributions. Together with the optimal states, covariance matrices representing confidence about the states are estimated. The main drawback of the Kalman filtering methods is that, because of the large dimensionality of SST images, estimating the covariance matrix of the states tends to be computationally expensive, as it requires the inversion of very large matrices.

In this paper, we propose a method based on the variational data assimilation framework ([2]) to assimilate missing data in SST images. Mainly, variational data assimilation has two advantages with respect to Kalman filtering techniques. First, it naturally deals with nonlinear state dynamics, and observation operator. Secondly, it does not require the estimation of the covariance matrix to measure the confidence about the optimal states, thus reducing the computational cost of the estimation. We proposed two types of methods. The first type is based on the minimization of a cost function involving a term about the observations composed of the observed SST image with missing data, and a term about the regularity of the final estimate. The second type of method is based on the minimization of a cost function composed of a term involving an entire sequence of SST images and a term about the temporal regularity of the sequence of desired outputs. Section 2 presents the methodology we developed for missing data interpolation, Section 3 provides preliminary results.

2. METHODOLOGY

2.1. Static image assimilation

We assume available an observation $\mathbf{Y}$ which consist only in the raw SST image without the missing data. This observation can be modeled as $\mathbf{Y} = \mathbf{P}\mathbf{X} + \epsilon$ where $\mathbf{X}$ is the SST image hidden state, $\mathbf{P}$ is a projection operator which keeps identical non missing data and removes the missing data, and ϵ is a centered Gaussian noise with covariance matrix $\mathbf{R}$. The problem of missing data interpolation when only a single image is available can be stated as finding the optimal state $\hat{\mathbf{X}}$ verifying:

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}} \mathbf{E}(\mathbf{X}) \text{ with } \mathbf{E}(\mathbf{X}) = |\mathbf{Y} - \mathbf{P}\mathbf{X}|^2_{\mathbf{R}^{-1}} + \mathbf{E}_R(\mathbf{X}) \quad (1)$$

where $|.|_{\mathbf{R}^{-1}}$ is the Mahalanobis distance associated with the matrix $\mathbf{R}^{-1}$ that can be written using its associated inner product as $|x|^2_{\mathbf{R}^{-1}} =< \mathbf{R}^{-1}x, x >$. $\mathbf{E}_R(\mathbf{X})$ is a cost function specifying the type of regularity expected on the hidden state. In this paper we tried two possible functions. First we set $\mathbf{E}_R(\mathbf{X}) = |\nabla\mathbf{X}|^2$ to enforce smoothness to the solution $\hat{\mathbf{X}}$. Secondly we set $\mathbf{E}_R(\mathbf{X}) = |\nabla\mathbf{X}|$ to enforce regularity of the level lines of the solution. The minimization of the energy $\mathbf{E}(\mathbf{X})$ is achieved by a gradient descent. Starting from an initial condition $\mathbf{X}(0)$, it resorts to the Euler-Lagrange evolution equation associated with

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Eq. 1: $\partial_t \mathbf{X} = -\frac{\delta \mathbf{E}}{\delta \mathbf{X}}$. In the first case where $\mathbf{E}_R(\mathbf{X}) = |\nabla \mathbf{X}|^2$ the differential of the cost function $\mathbf{E}(\mathbf{X})$ with respect to the state can be written:

$$\frac{\delta \mathbf{E}}{\delta \mathbf{X}} = -\mathbf{P}^* \mathbf{R}^{-1}(\mathbf{Y} - \mathbf{P}\mathbf{X}) - \text{div}(\nabla \mathbf{X}) \tag{2}$$

where $\mathbf{P}^*$ is the adjoint of the observation operator, and div(.) is the divergence operator. In the second case when $\mathbf{E}_R(\mathbf{X}) = |\nabla \mathbf{X}|$ the cost function differential is:

$$\frac{\delta \mathbf{E}}{\delta \mathbf{X}} = -\mathbf{P}^* \mathbf{R}^{-1}(\mathbf{Y} - \mathbf{P}\mathbf{X}) - \frac{1}{2}\text{div}\left(\frac{\nabla \mathbf{X}}{|\nabla \mathbf{X}|}\right) \tag{3}$$

2.2. Image sequence assimilation

For a temporal sequence of SST images with missing data $\mathbf{Y}(t) = \mathbf{P}(t)\mathbf{X}(t) + \epsilon(t)$, $t \in [t_0, t_f]$, we assume that the temporal evolution of the hidden state is driven by the dynamical system:

$$\begin{cases} \partial_t \mathbf{X} + \mathbf{M}(\mathbf{X}) = \mathbf{w}(t) \\ \mathbf{X}(t_0) = \mathbf{X}(0) + \eta \end{cases} \tag{4}$$

The variational data assimilation problem for the sequence consists in estimating the sequence of hidden state minimizing the cost:

$$\mathbf{J}(\mathbf{X}) = \frac{1}{2}\int_{t_0}^{t_f} |\mathbf{Y}(t) - \mathbf{P}(t)\mathbf{X}(t)|^2_{\mathbf{R}(t)^{-1}} dt + \frac{1}{2}\int_{t_0}^{t_f} |\partial_t \mathbf{X}(t) + \mathbf{M}(\mathbf{X}(t))|^2_{\mathbf{Q}^{-1}} dt + \frac{1}{2}|\mathbf{X}(t_0) - \mathbf{X}(0)|^2_{\mathbf{B}^{-1}} \tag{5}$$

The estimation of the optimal state sequence requires to cancel the differential of the cost function w.r.t. the state variable $\frac{\delta \mathbf{J}}{\delta \mathbf{X}}$. The variational data assimilation framework provide an iterative scheme to cancel the gradient by introducing adjoint variables defined as $\lambda = \mathbf{Q}^{-1}(\partial_t \mathbf{X} + \mathbf{M}(\mathbf{X} + \beta d\mathbf{X}))$, the dynamical model linear tangent operator defined as $\partial_{\mathbf{X}} \mathbf{M} d\mathbf{X} = \lim_{\beta \to 0} \frac{\mathbf{M}(\mathbf{X} + \beta d\mathbf{X}) - \mathbf{M}(\mathbf{X})}{\beta}$, and $\partial_{\mathbf{X}} \mathbf{M}^*$ is the adjoint of the linear tangent operator. Bennet *et al* [3] showed that canceling the gradient $\frac{\delta \mathbf{J}}{d\mathbf{X}}$ can be achieved by a retrograde integration of an evolution model on the adjoint variables driven by the dynamical model linear tangent and taking into account the observations, followed by a forward integration of an evolution model on the state differential taking into account the adjoint variables. Fig.1 gives the iterative scheme to compute the optimal sequence. As in the previous section, we considered two options to model the state dynamics. In the first option, we model the state dynamics as the Laplacian $\mathbf{M}(\mathbf{X}) = \mathbf{\Delta X} = \text{div}(\nabla \mathbf{X})$. This model has the specificity to be linear, thus equal to its linear tangent, and is auto-adjoint. In the second option, we modeled the state dynamics as curvature $\mathbf{M}(\mathbf{X}) = \text{div}(\frac{\nabla \mathbf{X}}{|\nabla \mathbf{X}|})$. This dynamical model is nonlinear, thus, is not suited to a Kalman filtering framework. The linear tangent operator corresponding to the dynamical model is:

$$\partial_{\mathbf{X}} \mathbf{M}(d\mathbf{X}) = \frac{\mathbf{\Delta} d\mathbf{X}}{|\nabla \mathbf{X}|} - \frac{\mathbf{\Delta X}}{|\nabla \mathbf{X}|^3} \nabla \mathbf{X}^T \nabla d\mathbf{X} - \frac{1}{|\nabla \mathbf{X}|^3}(2\nabla \mathbf{X}^T \mathbf{H}(\mathbf{X}) \nabla d\mathbf{X} + \nabla \mathbf{X}^T \mathbf{H}(d\mathbf{X}) \nabla \mathbf{X}) + \frac{3}{|\nabla \mathbf{X}|^5}(\nabla \mathbf{X}^T \mathbf{H}(\mathbf{X}) \nabla \mathbf{X}) \nabla \mathbf{X}^T \nabla d\mathbf{X} \tag{6}$$

where $\mathbf{H}(.)$ is the Hessian operator. The adjoint of the linear tangent of the dynamical model can be written as

$$\partial_{\mathbf{X}} \mathbf{M}^*(d\mathbf{X}) = \frac{\mathbf{\Delta} d\mathbf{X}}{|\nabla \mathbf{X}|} + \frac{\mathbf{\Delta X}}{|\nabla \mathbf{X}|^3} \nabla \mathbf{X}^T \nabla d\mathbf{X} - \frac{1}{|\nabla \mathbf{X}|^3}(2\nabla \mathbf{X}^T \mathbf{H}(\mathbf{X}) \nabla d\mathbf{X} - \nabla \mathbf{X}^T \mathbf{H}(d\mathbf{X}) \nabla \mathbf{X}) - \frac{3}{|\nabla \mathbf{X}|^5}(\nabla \mathbf{X}^T \mathbf{H}(\mathbf{X}) \nabla \mathbf{X}) \nabla \mathbf{X}^T \nabla d\mathbf{X} \tag{7}$$

3. RESULTS

We evaluated our algorithms for static and dynamic SST image data interpolation using 50 images of the region of Malvinas. One image is recorded every day. The images are part of the 2008 AVHRR METOP recordings. We present results only for the two methods designed for still images, and the method for image sequence based on the Laplacian state dynamics. Fig2 gives sample results produced by the methods we propose for still image and image sequence assimilation. The second and the third rows gives the results for methods for the assimilation of still images. The results for these methods are very similar

1: starting from $\hat{\mathbf{X}}(t_0) = \mathbf{X}(0)$ perform a forward integration of the system $\partial_t \hat{\mathbf{X}} + \mathbf{M}(\mathbf{X}) = 0$

2: given $\hat{\mathbf{X}}$ compute the adjoint variables $\lambda(t)$ using the system:

$$\begin{cases} -\partial_t d\lambda + \partial_{\mathbf{X}} \mathbf{M}^*(\lambda) = \mathbf{P}^*(t)(\mathbf{R}^{-1}(t)(\mathbf{Y}(t) - \mathbf{P}(t)\mathbf{X}(t))) \\ d\lambda(t_f) = 0 \end{cases} \quad (8)$$

3: compute the initial cost differential $d\mathbf{X}(t_0) = \mathbf{B}\lambda(t_0)$

4: from the adjoint variables $\lambda(t)$ compute the gradient $d\mathbf{X}(t)$ from the initial conditions $d\mathbf{X}(t_0)$ from a forward integration of the system $\partial_t d\mathbf{X} + \partial_{\mathbf{X}} \mathbf{M} d\mathbf{X} = \mathbf{Q}^{-1}\lambda(t)$

5: update the state estimate $\hat{\mathbf{X}} = \hat{\mathbf{X}} + d\mathbf{X}$

6: go to step 2 et and loop until convergence.

Fig. 1. Variational data assimilation algorithm.

despite the differences in the choice of the regularity constraints $\mathbf{E}_R(\mathbf{X})$ over the final solution. A careful visual inspection of the results shows that qualitatively good results are obtained. Only, temporal regularity in the sequence of estimates is missing. As showed in the last row of Fig2, the data assimilation method for sequence achieves to introduce this temporal regularity at a cost of smoother estimates. Further experiments about the variational data assimilation method for image sequence with the state dynamics based on the curvature are being conducted. This dynamical model is interesting as, in theory, it generates a diffusion process that better preserves the image level lines than the Laplacian dynamics.

4. REFERENCES

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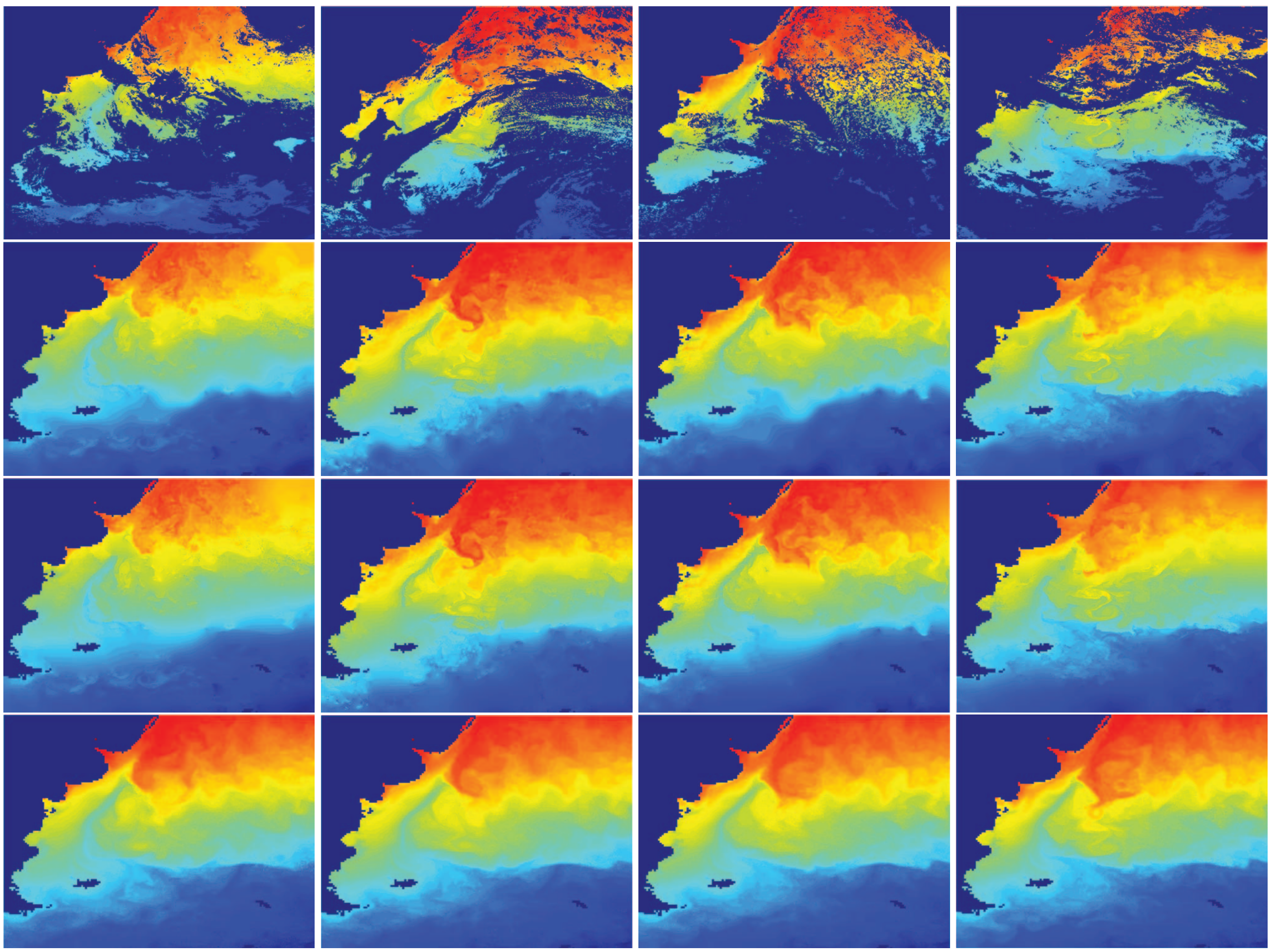

Fig. 2. Missing data interpolation sample results. In each row four images are given corresponding to the 10th, the 25th, the 35th, and the 50th day's SST image. The first row gives the raw images with missing data. The second row gives the interpolation results obtained the first static assimilation method (see Eq.2). The third row gives the results for the second static assimilation method (see Eq.3). The last row gives the results for the data assimilation for image sequences.