BLOCK-DIAGONAL REPRESENTATIONS FOR COVARIANCE-BASED ANOMALOUS CHANGE DETECTORS

Anna Matsekh and James Theiler

Space and Remote Sensing Sciences
Los Alamos National Laboratory
Los Alamos, NM 87545, USA

1. INTRODUCTION

Change detection methods are of crucial importance in many remote sensing applications such as monitoring and surveillance, where the goal is to identify and separate changes of interest from pervasive differences inevitably present in images taken at different times and in different environmental and illumination conditions. Anomalous change detection (ACD) methods aim to identify rare, unusual or anomalous changes [1]. A number of these algorithms can be expressed as quadratic functions of the data, where the coefficients are based on the covariances and cross-covariances of the two images [2]; among these methods are Chronochrome and Covariance Equalization [3], Multivariate Alteration Detection [4], Hyperbolic and Subpixel Hyperbolic methods [2, 5]. In the paper we will focus our attention on the RX, Hyperbolic, Chronochrome and Subpixel Hyperbolic methods in the formulation suggested in [2]. The knowledge of the eigenvalues of ACD matrices can provide valuable insights into the algebraic and numerical properties of the covariance-based quadratic ACD methods and may ultimately shed the light on the properties of the optimal ACD method. We will use singular vectors of the whitened cross-covariance matrix of two hyper-spectral images and the Golub-Kahan permutations [6, 7] for obtaining equivalent tridiagonal representations of the matrices of a family of covariance-based quadratic ACD methods. Due to the nature of the problem these tridiagonal matrices have block-diagonal structure, which we exploit in order to identify analytical expressions for the eigenvalues of ACD matrices as a function of the singular values of the whitened cross-covariance matrix. The block-diagonal structure of the matrices of the RX, Hyperbolic, Chronochrome and Subpixel Hyperbolic ACD revealed by the SVD and Golub-Kahan transformations shows both the similarities and the differences in the properties of these change detectors.

2. ANOMALOUS CHANGE DETECTORS IN BLOCK-DIAGONAL FORM

Consider two hyper-spectral images $D_x = [x_1, x_2, \ldots, x_N]^T$ and $D_y = [y_1, y_2, \ldots, y_N]^T$ where the pixels $x_i \in \mathbb{R}^{d_x}, i = 1, 2, \ldots, N$ in the D_x-image have d_x hyper-spectral channels and the pixels $y_i \in \mathbb{R}^{d_y}, i = 1, 2, \ldots, N$ in the D_y-image

This work was supported by the Los Alamos Laboratory Directed Research and Development (LDRD) program.
Los Alamos National Laboratory Unrestricted Release Number LA-UR 09-07941.

have d_y hyper-spectral channels. We can assume, without the loss of generality, that the pixels in the images D_x and D_y have zero mean. The scalar measure of anomalousness detected when comparing pixels $x_i \in D_x$ and $y_i \in D_y$ is defined as follows [2]

$$A(x_i, y_i) = (x_i^T \; y_i^T)\, Q\, (x_i \; y_i)^T, \tag{1}$$

where the specific form of the matrix $Q \in \mathbb{R}^{(d_x+d_y)\times(d_x+d_y)}$ is determined by the properties of the ACD method used. Q is a dense symmetric matrix that is a function of the cross covariance and covariance matrices of the two images D_x and D_y. The change between the pixels x_i and y_i is considered anomalous if $A(x_i, y_i)$ exceeds a given threshold. We define the cross covariance and the covariance matrices of the images D_x and D_y as follows: $X = D_x D_x^T/N$, $Y = D_y D_y^T/N$, $C = D_y D_x^T/N$. Covariance matrices X and Y are symmetric matrices of size $d_x \times d_x$ and $d_y \times d_y$ respectively, and the cross covariance matrix C is a rectangular $d_y \times d_x$ matrix. In the whitened coordinates [2] $\tilde{D_x} = X^{-1/2} D_x$, $\tilde{D_y} = Y^{-1/2} D_y$, that are used to 'normalize' the images with respect to illumination, environmental and other ubiquitous changes [1], the covariance and the cross covariance matrices take the following form $\tilde{X} = \tilde{D_x}\tilde{D_x}^T = I$, $\tilde{Y} = \tilde{D_y}\tilde{D_y}^T = I$, $\tilde{C} = \tilde{D_y}\tilde{D_x}^T = Y^{-1/2} C X^{-1/2}$. Consider singular value decomposition (SVD) of the whitened cross-covariance matrix $\tilde{C} = U\tilde{\Sigma}V^T$, where U and V are orthogonal matrices of singular vectors of the size $d_y \times d_y$ and $d_x \times d_x$ respectively and $\tilde{\Sigma}$ is a rectangular $d_y \times d_x$ matrix

$$\tilde{\Sigma} = \begin{cases} \begin{bmatrix} \Sigma & 0 \end{bmatrix}, \text{ if } d_y < d_x \\ \Sigma, \text{ if } d_y = d_x \\ \begin{bmatrix} \Sigma \\ 0 \end{bmatrix}, \text{ if } d_y > d_x \end{cases} \tag{2}$$

with the diagonal block $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ comprised of $n = \min\{d_y, d_x\}$ singular values $1 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. The fact that $\sigma_1 \leq 1$ follows immediately from the fact that due to the whitening the matrix is rescaled such that $\|\tilde{C}\|_2 \leq 1$. Since $\tilde{C} = Y^{-1/2} C X^{-1/2}$ and $\tilde{C} = U\tilde{\Sigma}V^T$ it follows that $\tilde{\Sigma} = U^T Y^{-1/2} C X^{-1/2} V$. Similar to the approach used in the Optimal Covariance Equalization method [8] we can transform already whitened images D_x and D_y in SVD basis as follows

$$\bar{x} = V^T \tilde{x}, \quad \bar{y} = U^T \tilde{y}. \tag{3}$$

Since U and V are orthogonal, transformation (3) preserves the eigenvalue spectra of the matrices providing an equivalent compact representation of the data that allows for efficient formulation and solution of quadratic covariance-based ACD problems. When expressed in SVD-transformed whitened coordinates (3) the matrix Q of the RX ACD [2] takes the following form

$$\tilde{Q}_{\mathrm{RX}} = \begin{pmatrix} I_{d_x} & \tilde{\Sigma} \\ \tilde{\Sigma} & I_{d_y} \end{pmatrix}^{-1} = \begin{pmatrix} I_n & \Sigma & 0 \\ \Sigma & I_n & 0 \\ 0 & 0 & I_m \end{pmatrix}^{-1}, \tag{4}$$

where $m = |d_x - d_y|$ and I_n is an identity matrix of the order n and $\tilde{\Sigma}$ is a rectangular block matrix defined in (2). Next we notice that there always exists an orthogonal permutation Π, $\Pi\Pi^T = I$ such that

$$T = \Pi \begin{pmatrix} I & \tilde{\Sigma} \\ \tilde{\Sigma} & I \end{pmatrix} \Pi^T = \Pi \begin{pmatrix} 0 & \tilde{\Sigma} \\ \tilde{\Sigma} & 0 \end{pmatrix} \Pi^T + I = \Pi J \Pi^T + I = G + I, \tag{5}$$

where J is the Jordan-Wielandt matrix [7] and G is its tridiagonal Golub-Kahan form [6, 7]. This means that T is a symmetric tridiagonal matrix with ones on the main diagonal and singular values of the whitened cross covariance matrix interlaced with zeros on the upper and low diagonals, that is, matrix T is block-diagonal with each block $i = 1, 2, \ldots, n$ of the form

$$\begin{pmatrix} 1 & \sigma_i \\ \sigma_i & 1 \end{pmatrix}. \tag{6}$$

We can now define the permuted RX matrix

$$\bar{Q}_{\text{RX}} = \Pi \tilde{Q}_{\text{RX}} \Pi^T = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}^{-1}, \tag{7}$$

where Π is the Golub-Kahan permutation (5). $\bar{Q}_{\text{RX}}$ is block-diagonal with its first n blocks $\bar{Q}_{\text{RX}_i}$, $i = 1, 2, \ldots, n$ of the form

$$\bar{Q}_{\text{RX}_i} = 1/(1 - \sigma_i^2) \begin{pmatrix} 1 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix} \tag{8}$$

followed by an identity block. Since $\bar{Q}_{\text{RX}}$ is a block-diagonal matrix its eigenvalue spectrum $\Lambda(\bar{Q}_{\text{RX}})$ is the union of the eigenvalue spectra of its blocks. It is easy to see that $\Lambda(\bar{Q}_{\text{RX}}) = \{1/(1 \pm \sigma_i), \underbrace{1, 1, \ldots, 1}_{m} \mid i = 1, \ldots, n\}$. We can obtain a family of covariance-based ACD matrices in the block-diagonal form similar to (8) by applying the SVD transformation (3) followed the Golub-Kahan tridiagonalization. There always exists a Golub-Kahan permutation Π, such, that SVD-transformed ACD matrices can be expressed in tridiagonal form. Similar to the RX case these tridiagonalized matrices can be viewed as block-diagonal matrices consisting of 2×2 blocks. Again, this simple block diagonal structure makes it easy to determine analytical expressions for the eigenvalues of these matrices as a function of the singular values of the whitened cross-covariance matrix of the data. In Table 1 we show the structure of the blocks and the eigenvalue spectra of SVD-transformed RX, Hyperbolic, Chronochrome and Subpixel Hyperbolic ACD matrices in Golub-Kahan-permuted form. Since $\sigma_i \in [0, 1]$, it is clear that the eigenvalues of the matrices of the RX and Chronochrome anomalous change detectors are positive, while the the eigenvalues of the Hyperbolic and Subpixel Hyperbolic ACD matrices may take both positive and negative values.

3. CONCLUSIONS

The presented methodology provides a novel approach for the analysis of the algebraic properties and for efficient numerical implementation of the RX, Hyperbolic, Chronochrome and Subpixel Hyperbolic ACD. Specifically, we demonstrated that it is sufficient to compute singular value decomposition of the cross covariance matrix of the data in whitened coordinates in order to almost immediately obtain highly structured sparse matrices of the RX, Hyperbolic,

Table 1. The block structure and the eigenvalues of the RX, Hyperbolic, Chronochrome and Subpixel Hyperbolic whitened ACD matrices in block-diagonal form.

ACD Matrix	Matrix Block Structure	Matrix Eigenvalues
$\bar{Q}_{\text{RX}}$	$1/(1-\sigma_i^2)\begin{pmatrix} 1 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix}$	$\{1/(1\pm\sigma_i),\ \underbrace{1,1,\ldots,1}_{m} \mid i=1,2,\ldots,n\}$
$\bar{Q}_{\text{Hyper}}$	$\sigma_i/(1-\sigma_i^2)\begin{pmatrix} \sigma_i & -1 \\ -1 & \sigma_i \end{pmatrix}$	$\{\mp\sigma_i/(1\pm\sigma_i),\ \underbrace{0,0,\ldots,0}_{m} \mid i=1,2,\ldots,n\}$
$\bar{Q}_{\text{CC}}$	$1/(1-\sigma_i^2)\begin{pmatrix} \sigma_i^2 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix}$	$\{(1+\sigma_i^2)/(1-\sigma_i^2),\ \underbrace{0,0,\ldots,0}_{n+m} \mid i=1,2,\ldots,n\}$
$\bar{Q}_{\text{Subpix}}$	$\sigma_i/(1-\sigma_i^2)^2\begin{pmatrix} -2\sigma_i & 1+\sigma_i^2 \\ 1+\sigma_i^2 & -2\sigma_i \end{pmatrix}$	$\{\pm\sigma_i/(1\pm\sigma_i)^2,\ \underbrace{0,0,\ldots,0}_{m} \mid i=1,2,\ldots,n\}$

Chronochrome and Subpixel Hyperbolic ACD and their eigenvalue spectra, significantly reducing matrix inversion costs while reformulating these methods in a compact, easy to analyze form.

4. REFERENCES

[1] M. T. Eismann, J. Meola, A. D. Stocker, S. G. Beaven, and A. P. Schaum, "Airborne hyperspectral detection of small changes," *Applied Optics*, vol. 47, no. 28, pp. F27–F45, 2008.

[2] J. Theiler, "Quantitative comparison of quadratic covariance-based anomalous change detectors," *Applied Optics*, vol. 47, no. 28, pp. F12–F26, 2008.

[3] A. P. Schaum and A. D. Stocker, "Linear chromodynamics models for hyperspectral target detection," in *Aerospace Conference Proceedings*. 2003, pp. 1879–1885, IEEE.

[4] A. A. Nielsen, "The regularized iteratively reweighted mad method for change detection in multi- and hyperspectral data," *IEEE Transactions on Image Processing*, vol. 16, no. 2, pp. 463–478, 2007.

[5] J. Theiler, "Subpixel anomalous change detection in remote sensing imagery," in *Proceedings of the IEEE Southwest Symposium on Image Analysis and Interpretation*. 2008, pp. 165–168, IEEE Computer Society.

[6] G. Golub and W. Kahan, "Calculating the singular values and pseudo-inverse of a matrix," *SIAM Journal on Numerical Analysis*, vol. 2, no. 2, pp. 205–224, 1965.

[7] K. V. Fernando, "Accurately counting singular values of bidiagonal matrices and eigenvalues of skew-symmetric tridiagonal matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 20, no. 2, pp. 373–399, 1998.

[8] A. Schaum and A. Stocker, "Estimating hyperspectral target signature evolution with a background chromodynamics model," in *International Symposium on Spectral Sensing Research*, 2003.