SIMULATION OF (M_1, M_2)-DEPENDENT RANDOM FIELDS WITH K-DISTRIBUTED MARGINALS

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ABSTRACT

A method to simulate a two-dimensional (m_1, m_2)-dependent random field $\mathbf{Y}$ with K-distributed marginals is presented. The simulation starts with a random field with independent and identically standardized normally distributed elements. Then a (m_1, m_2)-dependent matrix is calculated using weighted sums. It has identically standardized normally distributed marginals. From this matrix the desired random field is computed numerically.

***Index Terms*—** simulation, K-distributed marginals, random field, SAR

1. INTRODUCTION

During past years, many different probability density functions have been tried to describe the land clutter reflectivity statistics analytically, among them the log-normal and the Weibull pdf. In 1976, Jakeman and Pusey [1] introduced the K-distribution, originally for sea clutter, which since has found wide use also to describe the distribution of land clutter in synthetic aperture radar (SAR) images.

Therefore, the simulation of K-distributed random variables and random fields is important for example to investigate different estimators for the two parameters of K-distributed data. Blacknell [2] proposed a method to simulate correlated K-distributed clutter. But he restricts his presentation to correlations in one dimension and to small lags because of the complexity of the necessary calculations.

However, in a former analysis Kurz and Schimpf [3] explored SAR data regarding their spatial dependences. The main result was that even those data originating from a homogenous grassy area are not independent. For each investigated type of clutter dependences between adjacent pixels were found. In the case of single–look images, adjacent pixels may have a certain degree of correlation due to the way of SAR processing. Here, 'adjacent' means, that the two pixels have at least one common vertex. In the case of multi–look images, this mutual influence between adjacent pixels occurs with certainty.

Therefore, it is necessary to simulate two-dimensional random fields with K-distributed marginals which have a dependence structure that is not restricted to only one dimension.

2. MATHEMATICAL PRINCIPLES

2.1. About the K-Distribution

Let Z be the random amplitude (> 0) of an L-dimensional random vector, with a natural number L. In multi-look SAR data, the parameter L is two times the number of looks, because the amplitude of a pixel is calculated from $\frac{L}{2}$ complex random vectors. Below, L is treated as predetermined number.

Z is called K-distributed, if it obeys the following probability density function

$$f(x; a, \alpha) = \frac{x^{\alpha+L/2-1} a^{-\alpha-L/2}}{2^{\alpha+L/2-2}\Gamma(\alpha)\Gamma(L/2)} K_{\alpha-L/2}\left(\frac{x}{a}\right) \quad (1)$$

with parameters $a, \alpha > 0$ and the natural number L.

The distribution function corresponding to the density $f(\cdot; a, \alpha)$ is denoted by $\widetilde{K}(\cdot; a, \alpha)$.

Details on $K_{\alpha-L/2}$, the modified bessel function of the second kind, can be found in [4].

The mth moment, $m \geq 1$, of a K-distributed amplitude is

$$E(X^m) = a^m 2^m \frac{\Gamma(\frac{m}{2} + \alpha)\Gamma(\frac{m}{2} + \frac{L}{2})}{\Gamma(\alpha)\Gamma(\frac{L}{2})}. \quad (2)$$

2.2. (m_1, m_2)-Dependence

Consider the stochastic process

$$\{Y(k,l) | 1 \leq k \leq n_1, 1 \leq l \leq n_2\} = \{Y(k,l)\},$$

say, defined on a two-dimensional grid.

The stationary process $\{Y(k,l)\}$ is said to be (m_1, m_2)-dependent (where m_1, m_2 are non-negative integers) if for each (t, u) the two sets of random variables

$$\{Y(k,l) | 1 \leq k \leq t, 1 \leq l \leq u\}$$

and

$$\{Y(k,l) | t + m_1 + 1 \leq k \leq n_1 \text{ or } u + m_2 + 1 \leq l \leq n_2\}$$

are independent. This is illustrated in **Figure 1**. The set mentioned first includes the colored elements of the process, which are located in the upper left (orange). The second set is composed of the elements in the lower right (blue).

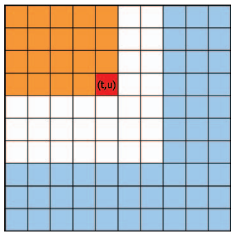

Fig. 1. Dependence structure of a $(3, 2)$-dependent process.

One possible description for (m_1, m_2)-dependent processes is as follows. The process Y is (m_1, m_2)-dependent, if and only if for indices fulfilling

$$|k - k'| > m_1 \text{ or } |l - l'| > m_2,$$

the two random variables $Y(k, l)$ and $Y(k', l')$ are independent.

3. SIMULATION OF (M_1, M_2)-DEPENDENT RANDOM FIELDS WITH K-DISTRIBUTED MARGINALS

A stationary random field $\mathbf{Y}$ will be simulated as a $N_1 \times N_2$ matrix. $\mathbf{Y}$ has to be stationary and (m_1, m_2)-dependent with identically $\widetilde{K}(\cdot; a, \alpha)$-distributed components. The parameters fulfill $a, \alpha > 0$, m_1, m_2 are nonnegative integers and N_1, N_2 are natural numbers. The elements of the matrix $\mathbf{Y}$ correspond to the points of a grid.

1. In the first step, a random field $\mathbf{N}$ with independent and identically standardized normally distributed components is simulated. $\mathbf{N}$ is a $(2N_1 + 2m_1 - 1) \times (2N_2 + 2m_2 - 1)$ matrix. The elements of $\mathbf{N}$ also correspond to the points of a grid. This is overlapping with the grid belonging to $\mathbf{Y}$ and has twice its resolution, see **Figure 2**.

   To simplify the notation, every matrix will be indexed on the finer grid. Hence it is

   $$\mathbf{N} = [N_{k,l}]_{\substack{k=1,\ldots,2N_1+2m_1-1 \\ l=1,\ldots,2N_2+2m_2-1}} \text{ and } \mathbf{Y} = [Y_{2k,2l}]_{\substack{k=1,\ldots,N_1 \\ l=1,\ldots,N_2}}.$$

   All the additionally needed rows or columns of $\mathbf{N}$ are added at the bottom or at the right side of the grid belonging to $\mathbf{Y}$. Otherwise the indices would depend on m_1 or m_2.

   In the following calculations the row-indices of $\mathbf{N}$ will be treated modulo $2N_1 + 2m_1 - 1$ and the column-indices of $\mathbf{N}$ modulo $2N_2 + 2m_2 - 1$. The row-indices of $\mathbf{Y}$ will be treated modulo $2N_1$ and the column-indices of $\mathbf{Y}$ modulo $2N_2$. To simplify the notation this is not reflected in the indices.

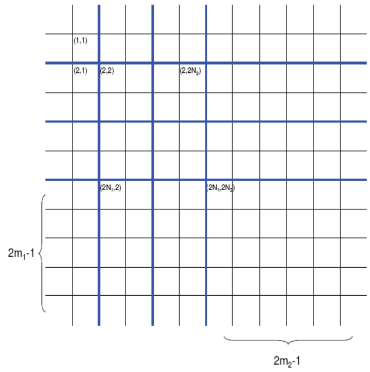

Fig. 2. Blue grid belonging to $\mathbf{X}$ and $\mathbf{Y}$, fine black grid belonging to $\mathbf{N}$.

2. In the second step the matrix $\mathbf{N}$ is used to calculate the matrix $\mathbf{X}$. It is (m_1, m_2)-dependent and its components correspond to the same grid as them of $\mathbf{Y}$.

   $$\mathbf{X} = [X_{2k,2l}]_{\substack{k=1,\ldots,N_1 \\ l=1,\ldots,N_2}}$$

   with the same rules of simplification for the indices as those of $\mathbf{Y}$.

   For the calculation of $\mathbf{X}$ it is necessary to choose coefficients $a_{k',l'}$ with $k' = -m_1, \ldots, m_1$ and $l' = -m_2, \ldots, m_2$. At least one of these coefficients has to be nonzero $a_{k',l'} \neq 0$.

   The elements of $\mathbf{X}$ are the weighted sums

   $$X_{2k,2l} = \left(\sum_{k'=-m_1}^{m_1} \sum_{l'=-m_2}^{m_2} a_{k',l'}^2 \right)^{-\frac{1}{2}} \cdot \sum_{k'=2k-m_1}^{2k+m_1} \sum_{l'=2l-m_2}^{2l+m_2} a_{k'-2k,l'-2l} N_{k',l'}$$
   $$\text{for} \quad k = 1, \ldots, N_1 \text{ and } l = 1, \ldots, N_2.$$

   This is illustrated in **Figure 3**.

   The elements of $\mathbf{X}$ are identically standardized normally distributed

   $$X_{2k,2l} \sim N(0, 1) \text{ for } k = 1, \ldots, N_1 \text{ and } l = 1, \ldots, N_2.$$

   They are even multivariate normally distributed because $\mathbf{X}$ is the linear image of the multivariate normally distributed matrix $\mathbf{N}$.

   Furthermore, $\mathbf{X}$ is (m_1, m_2)-dependent. Regarding two components $X_{2k,2l}$ and $X_{2\widetilde{k},2\widetilde{l}}$ with $|2k - 2\widetilde{k}| >$

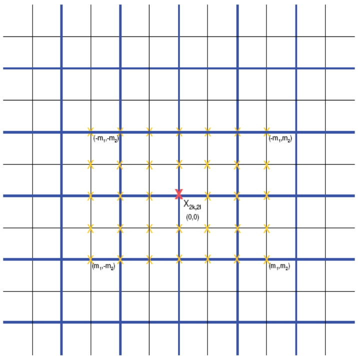

Fig. 3. $X_{2k,2l}$ (red) and the added components of $\mathbf{N}$ (yellow and red) weighted with the coefficients $a_{k',l'}$ with $k' = -m_1, \dots, m_1$ and $l' = -m_2, \dots, m_2$.

$2m_1$ or $|2l - 2\tilde{l}| > 2m_2$. They have a distance greater than m_1 in the direction of a row or greater than m_2 in the direction of a column regarded on the grid corresponding to $\mathbf{Y}$ and $\mathbf{X}$. $X_{2k,2l}$ and $X_{2\tilde{k},2\tilde{l}}$ are influenced by two disjoint blocks of elements of $\mathbf{N}$. The independence of these elements implies the independence of $X_{2k,2l}$ and $X_{2\tilde{k},2\tilde{l}}$.

3. In the third step a (m_1, m_2)-dependent matrix will be computed, whose elements are identically $\widetilde{K}(\cdot; a, \alpha)$ distributed.

   To obtain a $\widetilde{K}(\cdot; a, \alpha)$-distributed random variable Y a continuous random variable U is needed, which is uniformly distributed on the interval $(0, 1)$. From this it is possible to calculate the random variable Y using the quantile function $\widetilde{K}^{-1}(\cdot; a, \alpha)$ as follows

$$Y = \widetilde{K}^{-1}(U; a, \alpha). \tag{3}$$

   Because of the strict monotonic increase of $\widetilde{K}(\cdot; a, \alpha)$, equation (3) is equivalent to

$$\widetilde{K}(Y; a, \alpha) = U.$$

   Using the probability density function $f(\cdot; a, \alpha)$ defined in (1), it results

$$\int_0^Y f(u; a, \alpha) du = U. \tag{4}$$

The left side of equation (4) defines a function G

$$\begin{aligned} & G(y; a, \alpha) \\ = & \int_0^y f(u; a, \alpha) du \\ = & \frac{a^{-\alpha - L/2}}{2^{\alpha + L/2 - 2}\Gamma(\alpha)\Gamma(L/2)} \\ & \cdot \int_0^y u^{\alpha + L/2 - 1} K_{\alpha - L/2}\left(\frac{u}{a}\right) du \\ & \text{for } y > 0. \end{aligned}$$

Hence, Y is the solution of the equation

$$G(Y; a, \alpha) = U.$$

Therefore, each element of the matrix $\mathbf{Y}$ can be calculated from the corresponding element of the matrix $\mathbf{X}$ by solving the equation

$$G(Y_{2k,2l}; a, \alpha) = \Phi(X_{2k,2l}) \tag{5}$$
$$\text{for } k = 1, \dots, N_1 \text{ and } l = 1, \dots, N_2.$$

Here, Φ denotes the cumulative distribution function for the standardized normal distribution. $\Phi(X_{2k,2l}) \sim U(0, 1)$ and with the previous argumentation follows

$$\begin{aligned} & Y_{2k,2l} \sim \widetilde{K}(\cdot; a, \alpha) \\ & \text{for } k = 1, \dots, N_1 \text{ and } l = 1, \dots, N_2. \end{aligned}$$

The (m_1, m_2)-dependence of $\mathbf{X}$ is relayed to $\mathbf{Y}$, because the function Φ and the quantile function $\widetilde{K}^{-1}$ both are bijections. Therefore, the transformation (5) transforms independent random variables $X_{2k,2l}, X_{2\tilde{k},2\tilde{l}}$ into independent $Y_{2k,2l}, Y_{2\tilde{k},2\tilde{l}}$ and it transforms dependent random variables into dependent random variables.

4. SUMMARY AND OUTLOOK

In this paper a method to simulate a two-dimensional (m_1, m_2)-dependent random field $\mathbf{Y}$ with K-distributed marginals is presented.

First a random field $\mathbf{N}$ corresponding to a grid twice as fine as that of the actually desired random field is simulated. The elements of $\mathbf{N}$ are easy to compute, because they are independent and identically standardized normally distributed. Then a matrix $\mathbf{X}$ is calculated from $\mathbf{N}$ using weighted sums. $\mathbf{X}$ is defined on the desired grid and it has the predetermined dependence structure. From this the computation of $\mathbf{Y}$ is performed element by element as numerical solution of equation (5).

It has to be investigated how this approach could be transferred to other structures of dependence.

The influence of the weights $a_{k',l'}, k' = -m_1, \dots, m_1, l' = -m_2, \dots, m_2$ to the dependence between the elements of $\mathbf{Y}$ will be analyzed as well.

5. REFERENCES

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