DC Space-Charge Field Representation In Kisunko-Vainshtein Waveguide Excitation Theory

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***Abstract*—We obtain representations for static electric and magnetic fields induced by charged-particle current in a regular simply connected ideally conducting waveguide in the form of solutions to equations of the Kisunko-Vainshtein waveguide excitation theory.**

I. Introduction and Background

Usually, static ($\partial/\partial t = 0$, $\omega = 0$) electric and magnetic fields are detached from non-static ($\partial/\partial t \neq 0$, $\omega \neq 0$) ones [1] and treated separately by the method of Green's function [2] when one considers extended interaction of charged-particle (e.g. electron) beams with electromagnetic waves in the framework of the Kisunko-Vainshtein waveguide excitation theory [3], [4]. However, it may be also convenient to shape those static fields into the form of solutions to equations of the Kisunko-Vainshtein waveguide excitation theory itself and subsequently treat them on equal footing together with the non-static fields [5]. The rational for such a representation lies in the uniformity of utilization of expansion of the total excited electromagnetic field into the complete set of regular waveguide eigen-modes and realization of the fact that methods (see, e.g., [8]) developed for calculation of the dynamic (time-dependent) space-charge fields can be equally applied to their static counterparts (cf. [2]). Such a representation can be helpful in recently developed codes like the GPU-accelerated FDTD-PIC 'NEPTUNE' [6], [7] designed for simulations of state-of-the-art millimeter and sub-millimeter TWTs, etc.

II. Results

In the regular vacuum waveguide with ideally conducting walls excited by a charged-particle current (obeying the continuity equation), static ($\partial/\partial t = 0$, $\omega = 0$) electric and magnetic parts of the total electromagnetic field are subject to the equations (for the sake of visualization we assume the time periodic process and explicitly write the superscript "*p*" whenever specifying the potential part of electric field)

$$div\vec{E}_0^p(\vec{r}) = 4\pi\varrho_0(\vec{r}), \qquad rot\vec{E}_0^p(\vec{r}) = 0; \tag{1}$$

$$rot\vec{B}_0(\vec{r}) = 4\pi c^{-1}\vec{j}_0(\vec{r}), \;\; div\vec{B}_0(\vec{r}) = 0, \tag{2}$$

where the subscript "0" denotes the zero-order (static) time-Fourier components of the involved quantities ($div\vec{j}_0 = 0$).

The solution to system (2)

$$\vec{B}_0(\vec{r}) = \sum_q \Big\{ V_{0;\,q}^{\text{TE}\,|\,ev}(z)\vec{B}_{0;\,q}^{\text{TE}\,|\,ev}(\vec{r}) + V_{0;-q}^{\text{TE}\,|\,ev}(z)\vec{B}_{0;-q}^{\text{TE}\,|\,ev}(\vec{r}) \Big\} \tag{3}$$

can be obtained by a standard application of the Kisunko-Vainshtein waveguide excitation theory (cf., e.g., [4] and [8, Eqs. (9)-(10)] and notations therein)

$$\frac{dV_{0;\pm q}^{\text{TE}\,|\,ev}}{dz} = \pm \lim_{n\to 0} \frac{1}{N_{n;\,q}^{\text{TE}\,|\,ev}} \int_{S_\perp} (\vec{j}_0 \cdot \vec{E}_{n;\mp q}^{\text{TE}\,|\,ev})\, d^2\vec{r}_\perp. \tag{4}$$

where the superscripts explicitly indicate the relevant (evanescent) regular waveguide modes; the norm $N_{n;\,q}^{\text{TE}\,|\,ev}$ is given in [8]. This ensures that the limit on the right-hand side of (4) is finite (and real); also, mode $\text{TE}_{0;\,q}^{ev}$ is purely magnetostatic, [5].

Since the static electric field is potential (1), we expand $\vec{E}_0^p$ in the zero-order time-Fourier vector waveguide eigenfunctions of non-propagating (evanescent) TM-modes

$$\vec{E}_0^p(\vec{r}) = \sum_q \Big\{ V_{0;\,q}^{\text{TM}\,|\,ev}(z)\vec{E}_{0;\,q}^{\text{TM}\,|\,ev}(\vec{r}) + V_{0;-q}^{\text{TM}\,|\,ev}(z)\vec{E}_{0;-q}^{\text{TM}\,|\,ev}(\vec{r}) \Big\} \tag{5}$$

which, besides of being static and a complete set, do not have the magnetic field contribution. It then follows from the first equation in (1) that ($\vec{l}$ denotes the unit vector along the z-axis)

$$\sum_q \left\{ \frac{dV_{0;\,q}^{\text{TM}\,|\,ev}}{dz}(\vec{l}\cdot\vec{E}_{0;\,q}^{\text{TM}\,|\,ev}) + \frac{dV_{0;-q}^{\text{TM}\,|\,ev}}{dz}(\vec{l}\cdot\vec{E}_{0;-q}^{\text{TM}\,|\,ev}) \right\} = 4\pi\varrho_0 \tag{6}$$

since $\vec{\nabla}\cdot\vec{E}_{0;\pm q}^{\text{TM}|ev}(\vec{r}) = 0$ by modes construction and the known vector identity $\vec{\nabla}\cdot(f\vec{X}) = \vec{\nabla}f\cdot\vec{X} + f\vec{\nabla}\cdot\vec{X}$. The second equation in (1) also gives

$$\sum_q \left\{ \frac{dV_{0;\,q}^{\text{TM}\,|\,ev}}{dz}(\vec{l}\times\vec{E}_{0;\,q}^{\text{TM}\,|\,ev}) + \frac{dV_{0;-q}^{\text{TM}\,|\,ev}}{dz}(\vec{l}\times\vec{E}_{0;-q}^{\text{TM}\,|\,ev}) \right\} = 0, \tag{7}$$

where we took into account that $\vec{\nabla}\times\vec{E}_{n;\pm q}^{\text{TM}|ev} = -i(n\omega/c)\,\vec{B}_{n;\pm q}^{\text{TM}|ev}$ ($n = 0$) as well as the known vector identity $\vec{\nabla}\times(f\vec{X}) = \vec{\nabla}f\times\vec{X} + f\vec{\nabla}\times\vec{X}$. Firstly, we vector multiply (7) on the left by $\vec{E}_{0;\mp q'}^{\text{TM}|ev}$, use the vector identity $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ and then scalar multiply the result obtained by $\vec{l}$; secondly, we multiply (6) by $(\vec{l}\cdot\vec{E}_{0;\mp q'}^{\text{TM}|ev})$, respectively. Adding thus received pairs of scalar equations, we finally get the following result:

$$\sum_q \left\{ \frac{dV^{\mathrm{TM}\,|\,ev}_{0;\,q}}{dz} (\vec{E}^{\mathrm{TM}\,|\,ev}_{0;\mp q\,'} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\,q}) \; + \right.$$

$$\left. + \frac{dV^{\mathrm{TM}\,|\,ev}_{0;-q}}{dz} (\vec{E}^{\mathrm{TM}\,|\,ev}_{0;\mp q\,'} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;-q}) \right\} \;=\; 4\pi \varrho_0 (\vec{l} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\mp q\,'}). \quad (8)$$

Integrating (8) across the regular waveguide cross-section and taking into account that

$$\int\limits_{S_\perp} (\vec{E}^{\mathrm{TM}\,|\,ev}_{0;\mp q\,'} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\pm q}) d^2\vec{r}_\perp = 0,$$

$$\int\limits_{S_\perp} (\vec{E}^{\mathrm{TM}\,|\,ev}_{0;\pm q\,'} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\pm q}) d^2\vec{r}_\perp = 4\pi \delta_{q\,'q} e^{\mp 2 k^{\mathrm{TM}}_{\perp|\,q} z} N^{\mathrm{TM}\,|\,ev}_{0;\,q} \quad (9)$$

(e.g., for the regular rectangular waveguide one can find that $N^{\mathrm{TM}\,|\,ev}_{0;\,q} = ab/(8\pi)$) and $(\vec{l} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;-q}) = e^{2k^{\mathrm{TM}}_{\perp|q} z} (\vec{l} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\,q})$ one finds an analogue of the Kisunko-Vainshtein waveguide theory excitation equations (cf. [8, Eqs. (9)-(10)]) for the amplitudes in (5):

$$\frac{dV^{\mathrm{TM}\,|\,ev}_{0;\pm q}}{dz} = \frac{1}{N^{\mathrm{TM}\,|\,ev}_{0;\,q}} \int\limits_{S_\perp} \varrho_0 (\vec{l} \cdot \vec{E}^{\mathrm{TM}\,|\,ev}_{0;\mp q})\, d^2\vec{r}_\perp. \quad (10)$$

Note that the second equation in (9) also serves as the definition of the norm $N^{\mathrm{TM}\,|\,ev}_{0;\,q}$.

In compact notations, modes of a regular simply connected ideally conducting waveguide can be written as follows [9], [10] (upper and lower signs are held for the TE and TM modes, respectively; the asterisk * denotes complex conjugation): $\vec{E}^{\mathrm{TX}|pr}_{n;q}(\vec{r}) = \vec{e}^{\mathrm{TX}|pr}_{n;q}(\vec{r}_\perp) e^{ik^{\mathrm{TX}\,|pr}_{z|n;q} z}$, $\vec{B}^{\mathrm{TX}|pr}_{n;q}(\vec{r}) = \vec{b}^{\mathrm{TX}|pr}_{n;q}(\vec{r}_\perp) e^{ik^{\mathrm{TX}\,|pr}_{z|n;q} z}$, $\vec{E}^{\mathrm{TX}|pr}_{n;-q}(\vec{r}) \equiv \pm \left[\vec{E}^{\mathrm{TX}|pr}_{n;q}(\vec{r})\right]^*$, $\vec{B}^{\mathrm{TX}|pr}_{n;-q}(\vec{r}) \equiv \mp \left[\vec{B}^{\mathrm{TX}|pr}_{n;q}(\vec{r})\right]^*$; $\vec{E}^{\mathrm{TX}\,|\,ev}_{n;\,q}(\vec{r}) = \vec{e}^{\mathrm{TX}\,|\,ev}_{n;\,q}(\vec{r}_\perp) e^{-|k^{\mathrm{TX}\,|ev}_{z|n;q}|z}$, $\vec{B}^{\mathrm{TX}|ev}_{n;q}(\vec{r}) = \vec{b}^{\mathrm{TX}\,|ev}_{n;\,q}(\vec{r}_\perp) e^{-|k^{\mathrm{TX}|ev}_{z|\,n;\,q}|z}$, $\vec{E}^{\mathrm{TX}\,|ev}_{\perp|\,n;-q}(\vec{r}) \equiv \pm e^{2|k^{\mathrm{TX}\,|ev}_{z|\,n;\,q}|z} \vec{E}^{\mathrm{TX}\,|ev}_{\perp|\,n;\,q}(\vec{r})$, $\vec{B}^{\mathrm{TX}\,|\,ev}_{\perp|\,n;-q}(\vec{r}) \equiv \mp e^{2|k^{\mathrm{TX}\,|ev}_{z|\,n;\,q}|z} \vec{B}^{\mathrm{TX}\,|\,ev}_{\perp|\,n;\,q}(\vec{r})$, $E^{\mathrm{TX}\,|\,ev}_{z|\,n;-q}(\vec{r}) \equiv e^{2|k^{\mathrm{TX}\,|ev}_{z|\,n;\,q}|z} E^{\mathrm{TX}\,|\,ev}_{z|\,n;\,q}(\vec{r})$, $B^{\mathrm{TX}\,|\,ev}_{z|\,n;-q}(\vec{r}) \equiv e^{2|k^{\mathrm{TX}|ev}_{z|\,n;\,q}|z} B^{\mathrm{TX}\,|\,ev}_{z|\,n;\,q}(\vec{r})$;

$$\vec{e}^{\,\mathrm{TE}\,|\,t}_{n;\,q}(\vec{r}_\perp) = \frac{in\omega}{c(k^{\mathrm{TE}}_{\perp|\,q})^2} (\vec{\nabla}_\perp \times \psi_q \vec{l}),$$

$$\vec{b}^{\,\mathrm{TE}\,|\,t}_{n;\,q}(\vec{r}_\perp) = \frac{(\zeta^{\,\mathrm{TE}\,|\,t})'}{\zeta^{\,\mathrm{TE}\,|\,t}} \frac{1}{(k^{\mathrm{TE}}_{\perp|\,q})^2} \vec{\nabla}_\perp \psi_q + \psi_q \vec{l};$$

$$\vec{e}^{\,\mathrm{TM}\,|\,t}_{n;\,q}(\vec{r}_\perp) = \frac{(\zeta^{\,\mathrm{TM}\,|\,t})'}{\zeta^{\,\mathrm{TM}\,|\,t}} \frac{1}{(k^{\mathrm{TM}}_{\perp|\,q})^2} \vec{\nabla}_\perp \varphi_q + \varphi_q \vec{l},$$

$$\vec{b}^{\,\mathrm{TM}\,|\,t}_{n;\,q}(\vec{r}_\perp) = -\frac{in\omega}{c(k^{\mathrm{TM}}_{\perp|\,q})^2} (\vec{\nabla}_\perp \times \varphi_q \vec{l});$$

where for the propagating (superscript "*pr*", $q < q_n^{cr|\mathrm{TX}}$) and evanescent (superscript "*ev*", $q \geq q_n^{cr|\mathrm{TX}}$) modes ($t = \{pr, ev\}$, $\mathrm{X} = \{\mathrm{E}, \mathrm{M}\}$, $k^{\mathrm{TX}\,|\,ev}_{z\,|\,n;\,q} \equiv i|k^{\mathrm{TX}\,|\,ev}_{z\,|\,n;\,q}|$, the prime $'$ denotes differentiation in z) $\zeta^{\,\mathrm{TX}\,|\,pr} \equiv e^{ik^{\mathrm{TX}\,|\,pr}_{z\,|\,n;\,q} z}$ and $\zeta^{\,\mathrm{TX}\,|\,ev} \equiv e^{-|k^{\mathrm{TX}\,|\,ev}_{z\,|\,n;\,q}|z}$,

$$k^{\mathrm{TX}\,|\,pr}_{z\,|\,n;\,q} = \left[\left(\frac{n\omega}{c}\right)^2 - \left(k^{\mathrm{TX}}_{\perp\,|\,q}\right)^2 \right]^{1/2} \left(\frac{n\omega}{c} > k^{\mathrm{TX}}_{\perp\,|\,q}\right),$$

$$|k^{\mathrm{TX}\,|\,ev}_{z\,|\,n;\,q}| = \left[\left(k^{\mathrm{TX}}_{\perp\,|\,q}\right)^2 - \left(\frac{n\omega}{c}\right)^2 \right]^{1/2} \left(\frac{n\omega}{c} \leq k^{\mathrm{TX}}_{\perp\,|\,q}\right).$$

Here the scalar (dimensionless) eigen-functions, $\chi_q = \{\varphi_q, \psi_q\}$, of the transverse (2D) Laplace operator

$$\triangle_\perp \chi_q + \left(k^{\mathrm{TX}}_{\perp\,|\,q}\right)^2 \chi_q = 0$$

obey the Dirichlet, $\varphi_q|_{S_\perp} = 0$, and Neumann, $\partial\psi_q/\partial n|_{S_\perp} = 0$, boundary conditions, respectively, and $q \equiv \{m, l\}$ is the multiindex numbering the modes.

III. Conclusion

Evanescent (non-propagating) modes are employed to describe the static (time-independent) potential (electric) space-charge field of a charged-particle beam propagating in a regular simply connected ideally conducting waveguide in terms of equations of excitation similar to those of the Kisunko-Vainshtein theory. The dynamic (time-dependent) space-charge fields in these settings are treated like in [8]. Thus, total electromagnetic field excited by a charged-particle beam in a regular simply connected ideally conducting waveguide can be calculated by uniform modelling methods. It turns out that static and dynamic (time-dependent) potential (electric) parts of the space-charge field can be represented by the $\mathrm{TM}^{ev}_{0;\,q}$-modes, which are not "supported" by a space empty from charges (i.e. if $\varrho(\vec{r}, t) = 0$) in a regular simply connected waveguide. The dynamic part of the space-charge field is equivalent (through the continuity equation) to the divergence of (the potential part of) current density.

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